Existence and Stability of Uniform Attractors for N-Dimensional Impulsive-Perturbed Parabolic System

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Introduction and setting of the problem

The qualitative theory of differential equations with impulsive perturbations is outlined in [1,10,14], and for impulsive dynamical systems in [3,5,9,11,12]. In the case of an infinite dimensional phase space, the qualitative behavior of dissipative systems is studied in the framework of the theory of global attractors [15]. The generalization of the basic concepts and results of the theory of attractors to infinite-dimensional impulsive dynamical systems was carried out in [4,7,13]. The main object of research is the minimal compact uniformly attracting set – uniform attractor. The questions of existence, structure and invariance of uniform attractors for different classes of infinitely dimensional impulsive systems are dealt with in [4,6,7]. In [8], authors proposed the conditions for impulsive semiflows, which guarantee the stability of the non-impulsive part of uniform attractors. In the present paper, we refine these conditions and apply them to the study of the stability of uniform attractors of a weakly-nonlinear N-dimensional impulsive-perturbed parabolic system. More precisely, in bounded domain $\Omega\subset R^n,\ n\geq 1$ we consider the following N-dimensional weakly nonlinear parabolic system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=a_1\Delta u_1-\varepsilon f_1(u_1,\dots,u_N),\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \dfrac{\partial u_N}{\partial t}=a_N\Delta u_N-\varepsilon f_N(u_1,\dots,u_N),\\ u_1\big|_{\partial\Omega}=\dots=u_N\big|_{\partial\Omega}=0, \end{cases} \tag{1}$$

where $\varepsilon>0$ is a small parameter, $a_i>0,\ f=(f_1,\dots,f_N)^T$ is a nonlinear vector-function, $f\in C^1(R^2)$ satisfies

$$\exists\, C>0,\ \ \forall\, u\in R^N,\ \ \forall\, i=\overline{1,N}\quad |f_i(u)|\leq C,\ \ f'(u)\geq -C. \tag{2}$$

These assumptions guarantee global existence and uniqueness of a weak solution of the problem (1) for every initial data from the phase space $X=(L^2(\Omega))^N$ having the norm $\|u\|_X=\sqrt{\sum\limits_{i=1}^N\|u_i\|^2}$. (Here $\|\cdot\|$ and $(\cdot,\cdot)$ mean a norm and a scalar product in $L^2(\Omega)$.)

For fixed positive numbers $\alpha_1,\dots,\alpha_N$, μ and for the function $\psi\in L^2(\Omega)$ the following impulsive problem is considered: the phase point $u(t)$, when it encounters the impulse set

$$M=\Big\{u\in X\ |\ \sum_{i=1}^N\alpha_i(u_i,\psi)^2=1\Big\}, \tag{3}$$

is transferred to a new position $Iu \in M'$ using impulsive map $I : M \to M'$, where

$$M' = \Big\{u \in X \mid \sum_{i=1}^{N} \alpha_i (u_i, \psi)^2 = 1 + \mu\Big\}. \tag{4}$$

It is proved in the paper that, for a sufficiently wide class of impulsive mappings $I : M \to M'$, the impulsive-perturbed problem (1)–(4) generates an impulse semiflow for sufficiently small ε generates a pulsed semiflow G_ε, which has a uniform attractor Θ_ε having an invariant and stable non-impulsive part, provided that the impulsive mapping $I : M \to M'$ is continuous.

Existence and stability of the uniform attractor of impulsive systems

Let a continuous semigroup $V : R_+ \times X \to X$, the impulsive set $M \subset X$, and the impulsive mapping $I : M \to X$ be given in the phase space $(X, \| \cdot \|_X)$. The impulsive semiflow $G : R_+ \times X \to X$ is constructed according to the following rule: [9]: if $V(t,x) \notin M$ for $x \in X$ and for all $t > 0$, then $G(t,x) = V(t,x)$; otherwise

$$G(t,x) = \begin{cases} V(t - T_n, x_n^+), & t \in [T_n, T_{n+1}), \\ x_{n+1}^+, & t = T_{n+1}, \end{cases} \tag{5}$$

where $T_0 = 0$, $T_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x_{n+1}^+ = IV(s_n, x_n^+)$, $x_0^+ = x$, s_n are the intervals between moments of impulsive perturbations characterized by the condition $V(s_n, x_n^+) \in M$.

Under conditions

$$\begin{gathered} M\text{– closed}, \ \ M \cap IM = \varnothing, \\ \forall\, x \in M, \ \ \exists\, \tau = \tau(x) > 0, \ \ \forall\, t \in (0, \tau) \quad V(t,x) \notin M, \\ \forall\, x \in X \quad t \to G(t,x) \ \text{ defined on } [0, +\infty) \end{gathered} \tag{6}$$

the formula (5) determines a semigroup $G : R_+ \times X \to X$ [3, 7], which we will call an *impulsive semiflow*.

Remark 1. It follows from conditions (6) and the continuity of the V [3, 6] that for an arbitrary $x \in X$ or there exists a moment of the time $s := s(x) > 0$ such that $\forall\, t \in (0, s) \quad V(t,x) \notin M$, $V(s,x) \in M$, or $\forall\, t > 0 \quad V(t,x) \cap M = \varnothing$ (and in this case we put $s(x) = \infty$).

Definition ([7])**.** A compact $\Theta \subset X$ will be called a uniform attractor of the impulsive semiflow G if Θ is a uniformly attracting set, i.e., for any bounded $B \subset X$

$$\operatorname{dist}(G(t,B), \Theta) \longrightarrow 0, \ \ t \to \infty,$$

and Θ is minimal among all closed uniformly attracting sets.

Remark 2. A uniform attractor may not be invariant with respect to G [7].

Lemma 1. *Suppose that a continuous semigroup $V : R_+ \times X \to X$ and a map $I : M \to X$ satisfy the following conditions: there is a compactly embedded space $Y \Subset X$ such that*

$$\begin{aligned} \exists\, C_1 > 0, \ \ \exists\, \delta > 0, \ \ \forall\, t \ge 0, \ \ \forall\, x \in X \quad & \|V(t,x)\|_X \le \|x\|_X e^{-\delta t} + C_1, \\ \forall\, t > 0, \ \ \forall\, r > 0, \ \ \exists\, C(t,r) > 0, \ \ \forall\, x \quad & \|x\|_X \le r, \ \ \|V(t,x)\|_Y \le C(t,r), \\ \exists\, C_2 > 0, \ \ \forall\, x \in X \cap M \quad & \|Ix\|_X \le \|x\|_X + C_2, \\ \forall\, r > 0, \ \ \exists\, C(r) > 0, \ \ \forall\, x \in Y \cap M \quad & \|x\|_Y \le r, \ \ \|Ix\|_Y \le C(r), \\ \exists\, \overline{s} > 0, \ \ \forall\, x \in IM \quad & s(x) \ge \overline{s}. \end{aligned}$$

Then the impulsive semiflow G has an uniform attractor Θ.

It is known [2, 5] that one of the equivalent definitions of stability of a compact invariant set A with respect to a continuous semiflow is equality

$$A = D^+(A) := \bigcup_{x\in A} \big\{y \mid \ y = \lim G(t_n, x_n), \ x_n \to x, \ t_n \geq 0\big\}. \tag{7}$$

It was shown in [8] that a uniform attractor of an impulsive semiflow may not satisfy (7), although under additional assumptions regarding the nature of the behavior of the trajectories in the neighborhood of the impulsive set, we can obtain the following result.

Lemma 2 ([8]). *Let impulsive semiflow G has a uniform attractor Θ. Let impulsive mapping $I : M \to X$ be continuous, and the following conditions are satisfied:*

- *for any sequence $x_n \to x \in \Theta \setminus M$*

$$\begin{cases} s(x) = \infty, & \text{if } s(x_n) = \infty \text{ for infinitely many } n, \\ s(x_n) \to s(x), & \text{otherwise}; \end{cases}$$

- *for any sequence $x_n \to x \in \Theta \cap M$*

$$\text{either } s(x_n) = \infty \text{ for infinitely many } n, \text{ or } s(x_n) \to 0.$$

Then $\Theta \setminus M$ is invariant with respect to semiflow G and

$$\Theta = \overline{\Theta \setminus M}, \quad D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{8}$$

Application to impulsive-perturbed parabolic problem

To apply Lemmas 1, 2 to impulsive problems (1)–(4), we specify the perturbation parameters. Let $\{\lambda_k\}_{k=1}^{\infty}$, $\{\psi_k\}_{k=1}^{\infty}$ be solutions to the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$. Assume that in the definition of sets M, M' we have $\psi = \psi_1$, $\lambda = \lambda_1$. Then it is natural to consider the following class of impulsive mappings $I : M \mapsto M'$:

$$\text{for } \ u = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \psi_1 + \sum_{k=2}^{\infty} \begin{pmatrix} c_1^k \\ \vdots \\ c_N^k \end{pmatrix} \psi_k \in M \quad \text{we have } \ I(u) = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \psi_1 + \sum_{k=2}^{\infty} \begin{pmatrix} c_1^k \\ \vdots \\ c_N^k \end{pmatrix} \psi_k.$$

The simplest example: $\forall\, i = \overline{1, N} \quad d_i = \sqrt{1 + \mu}\, c_i$.

The main result of this paper is the following theorem.

Theorem. *Let conditions* (2) *be satisfied. Then for sufficiently small $\varepsilon > 0$, the problem* (1)–(4) *in the phase space $X = (L^2(\Omega))^N$ generates an impulsive semiflow having a uniform attractor Θ_ε. If, in addition, the map $I : M \mapsto M'$ is continuous, then Θ_ε has invariant non-impulsive part and satisfies the stability properties* (8).

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