# The Dirichlet Problem for Singular Two-Dimensional Linear Differential Systems

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We consider the two-dimensional linear differential system

$$u_i' = p_i(t)u_{3-i} + q_i(t) \quad (i = 1, 2) \tag{1}$$

with the boundary conditions

$$u_1(a+) = 0, \quad u_1(b-) = 0, \tag{2}$$

where $p_1$ and $q_1 : ]a, b[ \to \mathbb{R}$ are Lebesgue integrable functions, while the functions $p_2$ and $q_2 : ]a, b[ \to \mathbb{R}$ are Lebesgue integrable on every closed interval contained in $]a, b[$.

We are mainly interested in the case where the functions $p_2$ and $q_2$ have nonintegrable singularities at the points $a$ and $b$, i.e. the case, where

$$\int\limits_a^b \big(|p_2(t)| + |q_2(t)|\big)\,dt = +\infty.$$

System (1) is singular in that sense.

We have proved the theorem on the Fredholmity of problem (1), (2), and based on this theorem we have established unimprovable in a certain sense conditions guaranteeing the unique solvability of the above-mentioned problem. They are generalizations of some results by T. Kiguradze [1], concerning the unique solvability of the Dirichlet problem for singular second order linear differential equations.

We use the following notation.

$$[x]_+ = \frac{|x| + x}{2}\,, \quad [x]_- = \frac{|x| - x}{2}\,;$$

$u(t_0+)$ and $u(t_0-)$ are the right and the left limits, respectively, of the function $u$ at the point $t_0$;

$L([a, b])$ is the space of Lebesgue integrable on $[a, b]$ real functions;

$L_{loc}(]a, b[)$ is the space of real functions which are Lebesgue integrable on every closed interval contained in $]a, b[$;

If $p \in L([a, b])$, then

$$I_{a,b}(p)(t) = \int\limits_a^t p(s)\,ds \int\limits_t^b p(s)\,ds \ \text{ for } \ a \le t \le b.$$

A vector-function $(u_1, u_2) : ]a, b[ \to \mathbb{R}^2$ is said to be **a solution of system** (1) if its components are absolutely continuous on every closed interval contained in $]a, b[$ and satisfy system (1) almost everywhere on $]a, b[$.

A solution of system (1) satisfying the boundary conditions (2) is said to be **a solution of problem** (1)**,** (2).

Everywhere below it is assumed that

$$p_1 \in L([a,b]), \quad q_1 \in L([a,b]),$$
$$p_2 \in L_{loc}(]a,b[), \quad q_2 \in L_{loc}(]a,b[).$$

Along with system (1) we consider the corresponding homogeneous system

$$u_i' = p_i(t)u_{3-i} \quad (i=1,2). \tag{$1_0$}$$

**Theorem 1.** *Let the functions $p_1$ and $p_2$ satisfy the conditions*

$$p_1(t) \geq 0 \ \ \textit{for} \ \ a<t<b, \quad \delta = \int_a^b p_1(t)\,dt > 0, \tag{3}$$

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt < +\infty, \tag{4}$$

*and let the functions $q_1$ and $q_2$ satisfy the conditions*

$$\int_a^b I_{a,b}(p_1)(t)\big(I_{a,b}(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\big)\,dt < +\infty. \tag{5}$$

*If, moreover, the homogeneous problem $(1_0),(2)$ has only the trivial solution, then problem $(1),(2)$ has one and only one solution.*

**Remark 1.** If

$$\limsup_{t\to a+} \frac{p_1(t)}{(t-a)^{\alpha_0}} < +\infty, \quad \limsup_{t\to b-} \frac{p_1(t)}{(b-t)^{\beta_0}} < +\infty,$$
$$\limsup_{t\to a+} \frac{|q_1(t)|}{(t-a)^{\alpha_1}} < +\infty, \quad \limsup_{t\to b-} \frac{|q_1(t)|}{(b-t)^{\beta_1}} < +\infty,$$

where $\alpha_i > -1$, $\beta_i > -1$ $(i=0,1)$, then for conditions (4) and (5) to be satisfied it is sufficient that the conditions

$$\int_a^b (t-a)^{\alpha_0+1}(b-t)^{\beta_0+1}[p_2(t)]_-\,dt < +\infty,$$

$$\int_a^b \Big[(t-a)^{\alpha_0+\alpha_1+2}(b-t)^{\beta_0+\beta_1+2}\big[p_2(t)\big]_+ + (t-a)^{\alpha_0+1}(b-t)^{\beta_0+1}|q_2(t)|\,\Big]\,dt < +\infty$$

are fulfilled, respectively.

**Theorem 2.** *Let there exist a constant $\lambda \geq 1$ and a measurable function $p: ]a,b[ \to [0,+\infty[$ such that along with (3) the conditions*

$$[p_2(t)]_- = p(t)p_1^{1-\frac{1}{\lambda}}(t) \ \ \textit{for} \ \ a<t<b,$$
$$\int_a^b I_{a,b}(p_1)(t)p^{\lambda}(t)\,dt \leq \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta \tag{6}$$

*are satisfied. If, moreover, the functions $q_1$ and $q_2$ satisfy condition (5), then problem $(1),(2)$ has one and only one solution.*

**Corollary 1.** *If along with* (3) *and* (5) *the condition*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt \le \delta \tag{7}$$

*holds, then problem* (1), (2) *has one and only one solution.*

**Corollary 2.** *If along with* (3) *and* (5) *the conditions*

$$p_2(t) \ge -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t) \ \ \textit{for} \ \ a < t < b, \tag{8}$$

$$\operatorname{mes}\Big\{t \in ]a,b[:\ p_2(t) > -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)\Big\} > 0 \tag{9}$$

*hold, then problem* (1), (2) *has one and only one solution.*

**Remark 2.** Inequalities (6) and (7) in Theorem 2 and Corollary 1 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_{a,b}(p_1)(t)p^{\lambda}(t)\,dt \le \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta + \varepsilon$$

and

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt \le \delta + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

**Remark 3.** Inequalities (8) and (9) in Corollary 2 are unimprovable as well since if along with (3) and (5) the condition

$$p_2(t) \equiv -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)$$

holds, then problem (1), (2) either has no solution or has an infinite set of solutions.

**Remark 4.** Under the conditions of the above-formulated theorems and their corollaries, the function $p_2$ may have singularities of arbitrary order. For example, if

$$p_1(t) \equiv (t-a)^{\alpha}(b-t)^{\beta}, \quad p_2(t) \equiv \exp\Big(\frac{1}{(t-a)(b-t)}\Big),$$

$$|q_1(t)| \le (t-a)^{-2}(b-t)^{-2}\exp\Big(-\frac{1}{(t-a)(b-t)}\Big), \quad |q_2(t)| \le (t-a)^{\alpha_1}(b-t)^{\beta_1},$$

where $\alpha > -1$, $\beta > -1$, $\alpha_1 > -\alpha-2$, $\beta_1 > -\beta-2$, then the conditions of Theorems 1 and 2 as well as the conditions of Corollaries 1 and 2 are satisfied, and therefore problem (1), (2) has a unique solution.

# References

[1] T. Kiguradze, On solvability and unique solvability of two-point singular boundary value problems. *Nonlinear Anal.* **71** (2009), no. 3-4, 789–798.