Disconjugacy and Solvability of Dirichlet BVP for the Fourth Order Ordinary Differential Equations

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Consider on the interval $I=[a,b]$ the fourth order homogeneous linear ordinary differential equations

$$u^{(4)}(t)=p(t)u(t)-\mu q(t)u(t), \tag{0.1}$$

$$u^{(4)}(t)=p(t)u(t), \tag{0.2}$$

and the nonlinear equation

$$u^{(4)}(t)=p(t)u(t)+f(t,u(t))+h(t), \tag{0.3}$$

under the boundary conditions

$$u^{(i)}(a)=0, \quad u^{(i)}(b)=0 \ \ (i=0,1), \tag{0.4}$$

where $\mu\in R$, $h\in L(I,R)$, $p,q\in L(I,R_0^+)$, and $f\in K(I\times R,R)$. The study of the fourth order boundary value problems has increased recently, among them because they appear as a model equations for a large class of higher order parabolic equations arising, for instance, in statistical mechanics, phase field models, hydrodynamics, suspension bridges models, etc.

In [6] (see Lemma 4.2) it has been shown that the disconjugacy character of equation (0.1) implies the nonnegativity of Greens's function of problem $(0.1),(0.4)$. However, as we can see in [3], there are coefficients of (0.1), for which Green's function has constant sign but equation (0.1) is not disconjugate on I. For these reasons, we study disconjugacy of equation (0.1) on the interval I in connection with parameter μ, under the assumption that problem $(0.2),(0.4)$ has constant sign nonzero solution (see Definition 0.2). Also we find the necessary and sufficient conditions of nonnegativity of Green's function of problem $(0.1),(0.4)$ when $p\in D(I)$, and on the basis of these results we prove the sufficient conditions of solvability and unique solvability of the nonlinear problem $(0.3),(0.4)$.

The following notations are used throughout the paper.

- $R=]-\infty,+\infty[\,$, $R^+=]0,+\infty[\,$, $R_0^+=[0,+\infty[\,$;
- $C(I;R)$ is the Banach space of continuous functions $u:I\to R$ with the norm $\|u\|_C=\max\{|u(t)|:\ t\in I\}$;
- $\widetilde{C}^3(I;R)$ is the set of functions $u:I\to R$ which are absolutely continuous together with their third derivatives;

- $L(I;R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$;

- $K(I \times R;R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions.

By a solution of equation (0.3) we understand a function $u \in \widetilde{C}^3(I,R)$ which satisfies equation (0.3) a.e. on I.

Definition 0.1. Equation (0.1) is said to be disconjugate on I if every nontrivial solution u has less than four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 0.2. We say that $p \in D(I)$ if $p \in L(I;R_0^+)$, and problem (0.2), (0.4) has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,. \tag{0.5}$$

If we consider the equation

$$u^{(4)}(t) = \lambda p(t)u(t), \tag{0.6}$$

the set $D(I)$ can be interpreted as a set of the functions $p \in L(I,R_0^+)$ for which $\lambda = 1$ is the first eigenvalue of problem (0.6), (0.4).

1 Disconjugacy of equation (0.1)

Theorem 1.1. *Let $p \in D(I)$, $q \in L(I,R_0^+)$, $q \not\equiv 0$, and*

$$\mu_1 = \sup\big\{\mu : \ \ p(t) - \mu q(t) \geq 0 \ \text{ a.e. on } I\big\} > 0. \tag{1.1}$$

Then for an arbitrary $\mu \in \,]0,\mu_1]$ equation (0.1) is disconjugate on I.

Remark 1.1. Notice that condition (1.1) holds iff

$$\operatorname{mes}\big\{t \in I : \ p(t) = 0, \ q(t) \neq 0\big\} = 0.$$

Corollary 1.1. *Let $p_0 \in D(I)$, $p \in L(I,R)$, $p \not\equiv p_0$, and*

$$0 \leq p(t) \leq p_0(t) \ \text{ a.e. on } I. \tag{1.2}$$

Equation (0.2) is disconjugate on I.

From the last Corollary it immediately follows

Corollary 1.2. *Let $\lambda_1 \in R^+$ be such that $\lambda_1 p \in D(I)$. Then equation (0.6) is disconjugate on I for an arbitrary $\lambda \in [0,\lambda_1[\,$.*

Corollary 1.2 for $p \equiv 1$ is proved in [7] (see Theorem 3.1) and is optimal.

Remark 1.2 ([6, Lemma 4.2])**.** If equation (0.1) is disconjugate on I, then problem (0.1), (0.4) has only the trivial solution and its Green's function is nonnegative on $I \times I$.

2 Nonnegativity of Green's function of problem (0.1), (0.4)

The disconjugacy is only a sufficient condition in order to ensure the constant sign of Green's function of problem (0.1), (0.4). Now we give the theorem where necessary and sufficient conditions of nonnegativity of Green's function of problem (0.1), (0.4) are given when $p \in D(I)$, and $q \equiv 1$. Consider for this the boundary conditions

$$u(a) = \cdots = u^{(k-1)}(a) = 0, \quad u(b) = \cdots = u^{(3-k)}(b) = 0, \tag{2.1$_k$}$$

and let

$$\mu_2 = \min\{\mu_1', \mu_3'\},$$

where μ_k' $(k = 1, 3)$ are the least positive eigenvalues of problem (0.1), (2.1$_k$) (The existence of μ_1' and μ_3' for $q \equiv 1$ follows from the prove of Theorem 2.1). Then the next theorem is true.

Theorem 2.1. *Let $p \in D(I) \cap C(I, R^+)$, and $q \equiv 1$. Then problem* (0.1), (0.4) *has only the zero solution and its Green's function is nonnegative on $I \times I$ if and only if $\mu \in \,]0, \mu_2]$.*

3 Nonlinear problem

Theorem 3.1. *Let problem* (0.2), (0.4) *be uniquely solvable, its Green's function be nonnegative on $I \times I$, and the condition*

$$f(t, x) \operatorname{sign} x \leq \delta(t, x) \ \text{ for } \ |x| > r, \ \ t \in I, \tag{3.1}$$

hold, where $r \in R^+$, $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument, and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \delta(s, \rho)\, ds = 0. \tag{3.2}$$

Then problem (0.3), (0.4) *has at least one solution.*

From the last theorem by Remark 1.2 and Theorems 1.1, 2.1 it immediately follow.

Corollary 3.1. *Let $p \in D(I)$, $q \in L(I, R_0^+)$, $q \not\equiv 0$, and condition* (1.1) *hold. Let, moreover, conditions* (3.1) *and* (3.2) *be fulfilled, where $r \in R^+$, and $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument. Then the equation*

$$u^{(4)}(t) = (p(t) - \mu q(t))u(t) + f(t, u(t)) + h(t) \tag{3.3}$$

under the boundary conditions (0.4) *has at least one solution for an arbitrary $\mu \in \,]0, \mu_1]$.*

Corollary 3.2. *Let $p \in D(I) \cap C(I, R^+)$, $q \equiv 1$, and μ_2 be the constant defined in Theorem 2.1. Let, moreover, conditions* (3.1) *and* (3.2) *be fulfilled, where $r \in R^+$, and $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument. Then problem* (3.3), (0.4) *has at least one solution for an arbitrary $\mu \in \,]0, \mu_2]$.*

Theorem 3.2. *Let $p_0 \in D(I)$, $p \in L(I, R)$, $p \not\equiv p_0$,*

$$0 \leq p(t) \leq p_0(t) \ \text{ for } \ t \in I, \tag{3.4}$$

and the condition

$$-p(t)|x_1 - x_2| \leq (f(t, x_1) - f(t, x_2)) \operatorname{sign}(x_1 - x_2) \leq 0 \tag{3.5}$$

hold on $I \times R$. Then problem (0.3), (0.4) *is uniquely solvable.*

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