On Some Fine Properties of Supercritical Sigma-Perturbations

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A such that $\|A(t)\| \leq M < +\infty$ for all $t \geq 0$. We denote the Cauchy matrix of (1) by X_A and the highest Lyapunov exponent of (1) by $\lambda_n(A)$. Together with system (1) consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with piecewise continuous and bounded perturbation matrix Q such that

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \ \ t \geq 0. \tag{3}$$

Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

Let $\mathfrak{M}_\sigma(A)$ be the set of all perturbations Q satisfying condition (3) and having the appropriate dimensions. Any $Q \in \mathfrak{M}_\sigma$ is said to be a sigma-perturbation and the number $\nabla_\sigma(A) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}_\sigma(A)\}$ is called [7], [10, p. 225], [9, p. 214] the highest sigma-exponent or the Izobov exponent of system (1). It was proved in [7] that the Izobov exponent can be evaluated by means of the following algorithm:

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}} \frac{\xi_m(\sigma)}{m}\,, \tag{4}$$
$$\xi_m(\sigma) = \max_{k<m} \big(\ln \|X_A(m,k)\| + \xi_k(\sigma) - \sigma k\big), \ \ \xi_1 = 0, \ \ k \in \mathbb{N}.$$

According to [1,11], there exists a unique critical value $\sigma_0(A) \geq 0$ such that $\nabla_\sigma(A) = \lambda_n(A)$ for all $\sigma \geq \sigma_0(A)$ and $\nabla_\sigma(A) > \lambda_n(A)$ when $0 < \sigma < \sigma_0(A)$. It is well known that $\nabla_\sigma(A) = \lambda_n(A)$ for all $\sigma > 2M$ and, therefore, $\sigma_0(A) \leq 2M$. Using the Lyapunov $\sigma_{\mathrm{L}}(A)$, Grobman $\sigma_{\mathrm{G}}(A)$ or Perron $\sigma_{\mathrm{P}}(A)$ irregularity coefficients [4, pp. 67, 73], [8, pp. 77, 81] one can obtain some more accurate estimates for $\sigma_0(A)$. Indeed, the inequalities $\sigma_0(A) \leq \sigma_{\mathrm{L}}(A)$ and $\sigma_0(A) \leq \sigma_{\mathrm{G}}(A)$ were proved in [3] and [5]. It was also proved that the inequality $\sigma_0(A) \leq \sigma_{\mathrm{P}}(A)$ holds for $n = 2$, see [6], and is not valid for $n > 2$, see [12, 15]. These relations are combined in [15], where the irregularity quantity $\sigma_\lambda(A)$ is constructed in such a way that $\sigma_{\mathrm{G}}(A) \geq \sigma_\lambda(A) \geq \sigma_0(A)$ for all $n \in \mathbb{N}$ and $\sigma_\lambda(A) = \sigma_{\mathrm{P}}(A)$ for $n = 2$.

In [13] we give an explicit formula for evaluation of $\sigma_0(A)$ from the Cauchy matrix X_A of the original system. To formulate this result we need some notation.

Let $\mathcal{D}(m)$ be the set of all nonempty $d \subset \{1, \dots, m-1\} \subset \mathbb{N}$. Further we assume that for each $d \in \mathcal{D}(m)$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \cdots < d_s$ and $d = \{d_1, d_2, \dots, d_s\}$, where $s = |d|$ is the number of elements of the set d. We also put $\|d\| := d_1 + \cdots + d_s$ for $d \in \mathcal{D}(m)$ and $\|d\| := 0$ for $d = \varnothing$. In addition, for the sake of convenience we assume that $d_0 = 0$ and $d_{s+1} = m$ for each $d \in \mathcal{D}_0(m) := \mathcal{D}(m) \cup \{\varnothing\}$. Note that we do not include these additional elements in the set d. Under the above assumptions, let us define the quantity $\Xi(m,d)$ as

$$\Xi(m,d) := \sum_{i=0}^{s} \ln \|X_A(d_{i+1},d_i)\|,$$

where $m \in \mathbb{N}$, $d \in \mathcal{D}(m)$ and $s := |d|$. From [2,14] we can assert that

$$\xi_m(\sigma) = \max_{d \in \mathcal{D}_0(m)} \big(\Xi(m,d) - \sigma\|d\|\big). \tag{5}$$

Theorem 1 ([13])**.** *The equality*

$$\sigma_0(A) = \varlimsup_{m\to\infty} \max_{d \in \mathcal{D}(m)} \|d\|^{-1}(\Xi(m,d) - m\lambda_n(A)) \tag{6}$$

holds.

Theorem 2 ([13])**.** *The estimate*

$$\sigma_0(A) \geq \sigma^+ := \varlimsup_{m\to\infty} \max_{k<m} k^{-1}\big(\ln\|X_A(m,k)\| + \ln\|X_A(k,0)\| - m\lambda_n(A)\big) \tag{7}$$

is valid. If the limit $\lim\limits_{m\to\infty} m^{-1}\ln\|X_A(m,0)\|$ *exists, then* $\sigma_0(A) = \sigma^+$*.*

These theorems are obtained by direct inversion of (4) and (5) using some standard tools of convex analysis.

Since $\sigma_0(A)$ is said to be a critical value, we can say that all sigma-perturbations with $\sigma > \sigma_0(A)$ are supercritical. In order to investigate some fine properties of such perturbations we should modify the above expressions. It seems to be a natural idea to replace $m\lambda_n(A)$ by $\ln\|X_A(m,0)\|$ in (6) or (7). In this way we put

$$\sigma^{\#}(A) = \varlimsup_{m\to\infty} \max_{d \in \mathcal{D}(m)} \|d\|^{-1}\big(\Xi(m,d) - \ln\|X_A(m,0)\|\big).$$

Evidently, $\sigma^{\#}(A) \geq \sigma_0(A)$.

Let X_{A+Q} be the Cauchy matrix of system (2). Using the estimates for the norm of X_{A+Q} obtained in [14] we can prove the following statement.

Theorem 3. *If* $\sigma > \sigma^{\#}(A)$*, then* $\|X_{A+Q}(t,0)\| \leq K\|X_A(t,0)\|$ *with some* $K > 0$ *for all* $t > 0$*. If* $\sigma < \sigma^{\#}(A)$*, then* $\|X_{A+Q}(t,0)\|\|X_A(t,0)\|^{-1}$ *is unbounded as* $t \to +\infty$*.*

It should be noted that to reveal the meaning of

$$\sigma^{\Delta} := \varlimsup_{m\to\infty} \max_{k<m} k^{-1}\big(\ln\|X_A(m,k)\| + \ln\|X_A(k,0)\| - \ln\|X_A(m,0)\|\big)$$

still remains an open problem.

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