On Positive Periodic Solutions to Parameter-Dependent Second-Order Differential Equations with a Sub-Linear Non-Linearity

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We are interested in the existence and non-existence of a **positive** solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega).} \tag{0.1}$$

Here, $p, h, f \in L([0,\omega])$,

$$h(t) \geq 0 \ \text{ for a.e. } \ t \in [0,\omega], \ \ h(t) \not\equiv 0,$$

$\lambda \in]0,1[$, and a parameter $\mu \in \mathbb{R}$. By a *solution* to problem (0.1), as usually, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere, and verifies periodic conditions.

Definition 0.1. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \ \text{ for a.e. } \ t \in [0,\omega], \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \ \text{ for } \ t \in [0,\omega] \quad \big(\text{resp. } \ u(t) \leq 0 \ \text{ for } \ t \in [0,\omega]\big)$$

is fulfilled.

Definition 0.2. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.2}$$

has a positive solution.

For the cases $p \in \mathcal{V}^-(\omega)$, $p \in \mathcal{V}_0(\omega)$, and $p \in \mathcal{V}^+(\omega)$, we provide some results concerning the existence or non-existence of positive solutions to problem (0.1) depending on the choice of a parameter μ.

1 The case $p \in \mathcal{V}^-(\omega)$

Theorem 1.1. *Let $p \in \mathcal{V}^-(\omega)$ and*

$$\int_0^\omega [f(t)]_- \, dt > \exp\bigg(\int_0^\omega [p(t)]_+ \, dt\bigg) \int_0^\omega [f(t)]_+ \, dt.$$

Then there exists $\mu_* \geq 0$ such that

- *for any $\mu > \mu_*$, problem (0.1) has a unique positive solution,*
- *for any $\mu \leq \mu_*$, problem (0.1) has no positive solution.*

Theorem 1.1 yields immediately the following result.

Theorem 1.2. *Let $p \in \mathcal{V}^-(\omega)$ and*

$$\int_0^\omega [f(t)]_+ \, dt > \exp\bigg(\int_0^\omega [p(t)]_+ \, dt\bigg) \int_0^\omega [f(t)]_- \, dt.$$

Then there exists $\mu^* \leq 0$ such that

- *for any $\mu < \mu^*$, problem (0.1) has a unique positive solution,*
- *for any $\mu \geq \mu^*$, problem (0.1) has no positive solution.*

2 The case $p \in \mathcal{V}_0(\omega)$

Theorem 2.1. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int_0^\omega f(t)u_0(t) \, dt < 0,$$

where u_0 is a positive solution to problem (0.2). Then there exists $\mu_* \geq 0$ such that

- *for any $\mu > \mu_*$, problem (0.1) has a unique positive solution,*
- *for any $\mu \leq \mu_*$, problem (0.1) has no positive solution.*

From Theorem 2.1, we immediately derive the following result.

Theorem 2.2. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int_0^\omega f(t)u_0(t) \, dt > 0,$$

where u_0 is a positive solution to problem (0.2). Then there exists $\mu^* \leq 0$ such that

- *for any $\mu < \mu^*$, problem (0.1) has a unique positive solution,*
- *for any $\mu \geq \mu^*$, problem (0.1) has no positive solution.*

3 The case $p \in \mathcal{V}^+(\omega)$

Theorem 3.1. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{3.1}$$

be non-negative. Then there exists $-\infty < \mu_* < 0$ such that

- *for any $\mu > \mu_*$, problem* (0.1) *has a positive solution,*
- *for any $\mu < \mu_*$, problem* (0.1) *has no positive solution.*

Remark 3.1. The assumption about the non-negativity of u in Theorem 3.1 is meaningful. For instance, it follows from Definition 0.1 that the solution u to problem (3.1) is non-negative provided

$$f(t) \geq 0 \ \text{ for a.e. } \ t \in [0, \omega].$$

Moreover, it is known that if

$$\int\limits_0^\omega [f(t)]_+ \, \mathrm{d}t > \Delta(p) \int\limits_0^\omega [f(t)]_- \, \mathrm{d}t,$$

where $\Delta(p)$ is a number depending only on p, then the solution u to problem (3.1) is positive.

Theorem 3.1 yields immediately the following result.

Theorem 3.2. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

be non-positive. Then there exists $0 < \mu^* < +\infty$ such that

- *for any $\mu < \mu^*$, problem* (0.1) *has a positive solution,*
- *for any $\mu > \mu^*$, problem* (0.1) *has no positive solution.*

The last statement complements Theorems 3.1 and 3.2.

Theorem 3.3. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

change its sign. Then there exist $-\infty < \mu_* < 0$ and $0 < \mu^* < +\infty$ such that

- *for any $\mu \in]\mu_*, \mu^*[$, problem* (0.1) *has a positive solution,*
- *for any $\mu \in]-\infty, \mu_*[\cup]\mu^*, +\infty[$, problem* (0.1) *has no positive solution.*

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