Solution of Izobov–Bogdanov Problem on Irregularity Sets of Linear Differential Systems with a Parameter-Multiplier

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We consider depending on a parameter $\mu \in \mathbb{R}$ linear differential system

$$\dot{x} = \mu C(t)x, \;\; x(t) \in \mathbb{R}^n, \;\; t \geq 0 \tag{1_μ}$$

with a piecewise continuous bounded coefficients. By an irregularity set of the system

$$\dot{x} = C(t)x, \;\; x(t) \in \mathbb{R}^n, \;\; t \geq 0 \tag{2_C}$$

we call [2] the set of those values $\mu \in \mathbb{R}$ such that the corresponding system (1_μ) is irregular under Lyapunov.

E. K. Makarov constructed (see references in [2]) examples of systems (2_C) that have various metric and topological properties of their irregularity sets. Some of them have an arbitrary Lebesgue measure [5].

Later E. A.Barabanov proved [1] that every open set of real line without zero point can be realized as irregularity set of some system (2_C). Paper [4] held an analogous result for closed sets.

Recently P. A. Khudyakova has established that the reducibility sets of systems (1_μ) are exactly the class of F_σ sets [3].

In the present talk we completely describe the structure of irregularity sets for system (2_C), that solve N. A. Izobov's problem from [2].

For every $\varphi \in \mathbb{R}$ we denote a rotation matrix with the angle φ clockwise as

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

and let

$$J := U(2^{-1}\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For each $y = (y_1, y_2)^\top \in \mathbb{R}^2$ and 2×2-matrix Z we use the notations $\|y\| \equiv \sqrt{y_1^2 + y_2^2}$ for an Euclid norm and $\|Z\| \equiv \max\limits_{\|y\|=1} \|Zy\|$ for a spectral norm.

For any strongly increasing sequence $\{m_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ and for the numbers $5 \leq i_k \in \mathbb{N}$ we define the sequence $\{T_k\}_{k=1}^{+\infty}$, setting

$$T_1 := 2, \;\; T_{k+1} := m_k(i_k + 2)T_k, \;\; k \in \mathbb{N}.$$

Next let

$$\theta_k := m_k i_k T_k, \;\; \tau_k := \theta_k + m_k T_k, \;\; k \in \mathbb{N}.$$

For every sequence $\{b_k\}_{k=1}^{+\infty} \subset \mathbb{R}$ and for a number $d \in \mathbb{R}$, $d \neq 0$, we define the matrix $A(\,\cdot\,) = A(\,\cdot\,, d, \{m_k, i_k, b_k\}_{k=1}^{\infty})$, for each $l = \overline{1, T_k}$, $k \in \mathbb{N}$ setting

$$A(t) \equiv b_k J, \quad t \in (\tau_k - m_k l, \tau_k - m_k l + 1],$$
$$A(t) \equiv -b_k J, \quad t \in [\tau_k + m_k l - 1, \tau_k + m_k l).$$

For all other $t \geq 0$ let $A(t) \equiv d \operatorname{diag}[1, -1]$.

We denote as $X_A(t,s)$ the Cauchy matrix for system (2_A) and define the number $\delta(d)$ in the case $d > 0$ by the equality $\delta(d) := 1$, and in the case $d < 0$, let $\delta(d) := 2$. Let us denote as well

$$L_d(\alpha) := \Big\{ x \in \mathbb{R}^2 : \ \Big| \frac{x_{3-\delta(d)}}{x_{\delta(d)}} \Big| \leq \alpha \Big\}.$$

Note that

$$\begin{pmatrix} m & 0 \\ 0 & \dfrac{1}{m} \end{pmatrix} L_d(\alpha) = L_d(m^{-2 \operatorname{sgn} d} \alpha).$$

Lemma 1. *The matrix $X_A(T_{k+1}, \theta_k)$ is self-conjugated.*

For all $d \neq 0$ we define $k_0(d) \in \mathbb{N}$ by the equality $k_0(d) := 2 + [|d|^{-1}]$ ($[\,\cdot\,]$ denotes the integer part of a number).

Lemma 2. *For every $k \in \mathbb{N}$, $k \geq k_0(d) - 1$, the next inclusion holds*

$$X(T_{k+1}, T_{k_0(d)}) e_{\delta(d)} \subset L_d(2e^{4m_k T_k |d|}).$$

Let us denote

$$\widehat{Y}_\varkappa(\gamma) := U(\gamma) \operatorname{diag}[e^{\varkappa}, e^{-\varkappa}], \quad \gamma, \varkappa \in \mathbb{R}.$$

Lemma 3. *For all $\gamma, \varkappa \in \mathbb{R}$ such that $|\cos \gamma| \leq e^{-2|\varkappa|}$, the next estimation is true $\|\widehat{Y}^2_\varkappa(\gamma)\| < e^2$.*

Lemma 4. *If $d \neq 0$ and there exist $l \in \mathbb{N}$ and a sequence $(k_j)_{j=1}^{+\infty} \subset \mathbb{N}$ such that for all $p \in (k_j)_{j=1}^{+\infty}$ both the inequalities $i_p \leq l$, $m_p \geq 2 \max\{l, |d|^{-1}\}$ and the estimate $|\cos b_p| < e^{-2m_p|d|}$ hold, then system (2_A) is irregular under Lyapunov.*

Let us denote

$$\widetilde{L}_\varkappa := L_{\operatorname{sgn} \varkappa}(2^3 \varkappa^2), \quad \varkappa \in \mathbb{R}, \quad \widehat{L}_{k,d} := L_d(2^3 d^2 (m_k - 1)^2).$$

Lemma 5. *For all $\gamma, \varkappa \in \mathbb{R}$, $|\sin \gamma| \geq \varkappa^{-2}$, $\varkappa > 2^4$, the inclusion*

$$\widehat{Y}_\varkappa\Big(\gamma + \frac{\pi}{2}\Big) \widetilde{L}_\varkappa \subset \widetilde{L}_\varkappa$$

and for any $x \in \widetilde{L}_\varkappa$ the inequality

$$\Big\| \widehat{Y}_\varkappa\Big(\gamma + \frac{\pi}{2}\Big) x \Big\| > \|x\| e^{\varkappa - \sqrt{\varkappa}}$$

are correct.

Lemma 6. *For all $d \neq 0$, $k \in \mathbb{N}$ such that*

$$m_k > 1 + 2^4 |d|^{-1}, \quad |\cos b_k| \geq d^{-2} (m_k - 1)^{-2},$$

the inclusion

$$X_A(T_{k+1}, \theta_k - m_k + 1) \widehat{L}_{k,d} \subset \widehat{L}_{k,d}$$

holds, and for any solution $x(\,\cdot\,)$ of system (2_A) with the initial condition $x(\theta_k - m_k + 1) \in \widehat{L}_{k,d}$ for every $1 \leq l \leq 2T_k$ the next estimation is true

$$\frac{\|x(\theta_k + m_k l)\|}{\|x(\theta_k + m_k (l-1))\|} \geq e^{|d|(m_k - 1) - \sqrt{|d|(m_k - 1)}}.$$

Lemma 7. *If $m_k \to +\infty$ whereas $k \to +\infty$ and for any $l \in \mathbb{N}$ there exists $k_l \in \mathbb{N}$ such that for all $k \geq k_l$, satisfying the condition $i_k \leq l$, the estimate $|\cos b_k| > |d|^{-2}(m_k-1)^{-2}$ holds, then system (2_A) is regular under Lyapunov.*

Let M be an arbitrary $G_{\delta\sigma}$ set. One can find an open sets $\check{M}_{n,l} \subset \mathbb{R}$, $l,n \in \mathbb{N}$, for which the sets $\widetilde{M}_l$, $l \in \mathbb{N}$, defined by the equalities $\widetilde{M}_l := \bigcap\limits_{n=1}^{+\infty} \check{M}_{n,l}$, satisfy the relation $M = \bigcup\limits_{l=1}^{+\infty} \widetilde{M}_l$. Let us denote $\widehat{M}_{n,l} := \bigcap\limits_{p=1}^{n} \check{M}_{p,l}$. It is easy to see that the inclusion $\widehat{M}_{n+1,l} \subset \widehat{M}_{n,l}$ as well as the equality $\widetilde{M}_l = \bigcap\limits_{n=1}^{+\infty} \widehat{M}_{n,l}$ are correct.

We define by the recurrence a sequence $\{j_n\}_{n=0}^{\infty} \subset \mathbb{N} \cup \{0\}$, by set up

$$j_0 := 0, \ \ j_n := 2n\,9^{n+n^3} + j_{n-1}, \ \ n \in \mathbb{N}.$$

For any $k,l,n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we denote

$$J_n := \{j_{n-1}+1, \dots, j_n\}, \quad \varkappa_k(n) := 9^{-n-n^3}\big(k - 2^{-1}(j_n + j_{n-1})\big),$$
$$\rho_{n,l}(\alpha) = \rho_{n,l}(\alpha, \widehat{M}_{n,l}) := \inf_{\beta \in \mathbb{R} \setminus \widehat{M}_{n,l}} |\alpha - \beta|.$$

Moreover, let us denote $I_{n,k} = I_{n,k}(\{\widehat{M}_{n,l}\}_{n,l\in\mathbb{N}})$ for the set of all $l \in \mathbb{N}$ such that either $\rho_{n,l}(\varkappa_k(n)) \geq 2n^{-1}$, or there exists $p \in \{1, \dots, n-1\}$ for which

$$2n^{-1} \leq \rho_{p,l}(\varkappa_k(n)) \leq 5n^{-1}.$$

Lemma 8. *For all $\mu \notin M$ and $l \in \mathbb{N}$ one can find $n_0 = n_0(\mu, l) \in \mathbb{N}$ such that for every $n \geq n_0$ the correctness for some $k \in J_n$ of the inequality $|\mu - \varkappa_k(n)| < 2n^{-1}$ implies the inclusion $l \notin I_{n,k}$.*

For any integer k there exists a singular $n = n(k) \in \mathbb{N}$, for which $k \in J_n$. We define the values m_k, i_k and b_k, depending on a choice of the open sets $\check{M}_{n,l} \subset \mathbb{R}$, $l,n \in \mathbb{N}$, such that $M = \bigcup\limits_{l=1}^{+\infty} \bigcap\limits_{n=1}^{+\infty} \check{M}_{n,l}$, by the equalities

$$d := \mu, \ \ \mu \in \mathbb{R}, \ \ m_k := 1 + n(k)^2, \ \ n \in \mathbb{N}.$$

Let

$$i_k := \max\big\{5, \min I_{n,k}\big\}, \quad b_k(\mu) := 2^{-1}\pi + n^{-1}(\mu - \varkappa_k(n)), \ \ \mu \in \mathbb{R},$$

in the case $I_{n,k} \neq \varnothing$, and let

$$i_k := 5, \ \ b_k(\mu) \equiv 0, \ \ \text{if} \ \ I_{n,k} = \varnothing.$$

Let us define the matrix $\widetilde{A}_\mu(\,\cdot\,) = \widetilde{A}_\mu(\,\cdot\,, \{\widehat{M}_{n,l}\}_{n,l\in\mathbb{N}})$, $\mu \in \mathbb{R}$, by the equality

$$\widetilde{A}_\mu(t) := A(t) = A(t, d, \{m_k, i_k, b_k\}_{k=1}^{\infty}), \ \ t \geq 0,$$

with the defined as above values of parameters d, m_k, i_k, b_k.

Lemma 9. *If $0 \notin M$, then the system $(2_{\widetilde{A}_\mu})$ is irregular under Lyapunov for all $\mu \in M$ and is regular for any other $\mu \in \mathbb{R} \setminus M$.*

Let us denote by $\mathcal{T}$ the set of all $t \in \mathbb{R}_+ := \mathbb{R} \cap [0, +\infty)$ such that $\widetilde{A}_\mu(t) = \mu \operatorname{diag}[1, -1]$.

For any $t \in \mathcal{T}$ we define the function $\omega(\,\cdot\,)$ by the equality $\omega(t) \equiv 0$. For all other $t \in [T_k, T_{k+1})$, $k \in \mathbb{N}$, let $q_t := 0$ if $t < \tau_{k,j}$, and $q_t := 1$ in another case, and let $\omega(t) := (-1)^{q_t} b_k(0)$. We define a matrix $C(t)$, $t \geq 0$, by the relations

$$C(t) := U^{-1}(\tau)\Big(\widetilde{A}_1(t)U(\tau) - \frac{\mathrm{d}}{\mathrm{d}t}U(\tau)\Big), \quad t \geqslant 0, \quad \tau = \tau(t) := \int\limits_0^t \omega(s)\,ds. \tag{1}$$

Next statement contains the main result of this paper.

Theorem. *For every $G_{\delta\sigma}$ set $M \subset \mathbb{R}$, $0 \notin M$, system (1_μ) with the matrix $C(\,\cdot\,)$, given by equality* (1)*, is irregular under Lyapunov for all $\mu \in M$ and is regular for any other $\mu \in \mathbb{R}$.*

References

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