Dirichlet type Problem in a Smooth Convex Domain for Quasilinear Hyperbolic Equations of Fourth Order

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Let $\Omega=(0,\omega_1)\times(0,\omega_2)$ be an open rectangle, and let $\mathbf{D}$ be an *orthogonally convex* open domain with C^2 boundary inscribed in Ω such that

$$\begin{aligned}\mathbf{D}&=\{(x_1,x_2)\in\Omega:\ x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\}\\&=\big\{(x_1,x_2)\in\Omega:\ x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\},\end{aligned}$$

where $\gamma_i\in C([0,\omega_1])\cap C^2((0,\omega_1))$, $\eta_i\in C([0,\omega_2])\cap C^2((0,\omega_2))$ $(i=1,2)$, and

$$\gamma_1(\xi_1^*)=0,\quad \gamma_2(\xi_2^*)=\omega_2,\quad \eta_1(\zeta_1^*)=0,\quad \eta_2(\zeta_2^*)=\omega_1$$

for some $\xi_1^*,\xi_2^*\in[0,\omega_1]$ and $\zeta_1^*,\zeta_2^*\in[0,\omega_2]$.

In the domain $\mathbf{D}$ consider the problem

$$u^{(2,2)}=p_1(x_1,x_2)u^{(2,0)}+p_2(x_1,x_2)u^{(0,2)}+\sum_{j=0}^{1}\sum_{k=0}^{1}p_{jk}(x_1,x_2)u^{(j,k)}+q(x_1,x_2), \tag{1}$$

$$u(\eta_i(x_2),x_2)=\varphi_i(x_2)\ \ (i=1,2);\quad u^{(2,0)}(x_1,\gamma_i(x_1))=\psi_i''(x_1)\ \ (i=1,2), \tag{2}$$

where

$$u^{(j,k)}(x_1,x_2)=\frac{\partial^{j+k}u}{\partial x_1^j\partial x_2^k}\,,$$

$p_i\in C(\overline{\mathbf{D}})$ $(i=1,2)$, $p_{jk}\in C(\overline{\mathbf{D}})$ $(j,k=0,1)$, $q\in C(\overline{\mathbf{D}})$, $\phi_i\in C^2([0,\omega_2])$, $\psi_i\in C^2([0,\omega_1])$ $(i=1,2)$, $C^{m,n}(\overline{\mathbf{D}})$ is the Banach space of functions $u:\overline{\mathbf{D}}\to\mathbb{R}$, having continuous partial derivatives $u^{(i,j)}$ $(i=0,\dots,m;\ j=0,\dots,n)$, with the norm

$$\|u\|_{C^{m,n}(\overline{\mathbf{D}})}=\sum_{j=0}^{m}\sum_{k=0}^{n}\|u^{(j,k)}\|_{C(\overline{\mathbf{D}})},$$

and $\overline{\mathbf{D}}$ is the closure of the set $\mathbf{D}$.

Problem (1), (2) was studied in [1–3]. The Dirichlet problem for higher order linear hyperbolic equations in a rectangular domain was studied in [4].

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(2,2)}=p_1(x_1,x_2)u^{(2,0)}+p_2(x_1,x_2)u^{(0,2)}+\sum_{j=0}^{1}\sum_{k=1}^{1}p_{jk}(x_1,x_2)u^{(j,k)}, \tag{1_0}$$

$$u(\eta_i(x_2),x_2)=0\ \ (i=1,2);\quad u^{(2,0)}(x_1,\gamma_i(x_1))=0\ \ (i=1,2). \tag{2_0}$$

By a solution of problem (1), (2) we understand a *classical* solution, i.e., a function $u\in C^{2,2}(\mathbf{D})\cap C^{2,0}(\overline{\mathbf{D}})$ satisfying equation (1) and boundary conditions (2) everywhere in $\mathbf{D}$ and $\partial\mathbf{D}$, respectively.

Theorem 1. *Let* $p_i \in C(\overline{\Omega})$ $(i=1,2)$, $p_{jk} \in C(\overline{\Omega})$ $(j,k=0,1)$, $q \in C(\overline{\Omega})$, $\phi_i \in C^2([0,\omega_2])$, $\psi_i \in C^2([0,\omega_1])$ $(i=1,2)$, *and let*

$$p_1(x_1,x_2) \geq 0, \quad p_2(x_1,x_2) \geq 0 \;\; \textit{for} \;\; (x_1,x_2) \in \mathbf{D}.$$

Then problem (1), (2) *has the Fredholm property, i.e.:*

(i) *problem* $(1_0), (2_0)$ *has a finite dimensional space of solutions;*

(ii) *problem* (1), (2) *is uniquely solvable if and only if problem* $(1_0), (2_0)$ *has only the trivial solution.*

Furthermore, every solution of problem (1), (2) *in a unique way can be continued to a solution of equation* (1) *in the domain* Ω.

Remark 1. Orthogonal convexity of the domain D is very important and cannot be relaxed. Indeed, in the domain

$$D = \big\{(x_1,x_2): \ x_1 \in (0,4), \ x_2 \in (\gamma(x_1),2)\big\},$$

where

$$\gamma(x) = \begin{cases} e^{\frac{1}{(x-1)(x-3)}} & \text{for } \ x \in (1,3) \\ 0 & \text{for } \ x \in [0,1] \cup [3,4] \end{cases},$$

consider the problem

$$u^{(2,2)} = 0, \tag{3}$$

$$u\big|_{\partial \mathbf{D}} = 0; \quad u^{(2,0)}\big|_{\partial \mathbf{D}} = 1. \tag{4}$$

Notice that the function $y = \gamma(x)$ belongs to $C^\infty([0,4])$, it is increasing on the interval $[1,2]$ and it is decreasing on the interval $[2,3]$. It is easy to show that

$$\eta_1(y) = 2 - \sqrt{1 + \ln^{-1}(y)}$$

is the function inverse to $\gamma(x)$ on the interval $[1,2]$, and

$$\eta_2(y) = 2 + \sqrt{1 + \ln^{-1}(y)}$$

is the function inverse to $\gamma(x)$ on the interval $[2,3]$.

It is clear that the only possible solution of problem (3), (4) is a solution of the problem

$$u^{(2,0)} = 1, \tag{5}$$

$$u\big|_{\partial \mathbf{D}} = 0. \tag{6}$$

Problem (5), (6) has the unique solution

$$u(x_1,x_2) = \begin{cases} \dfrac{x_1(x_1 - \eta_1(x_2))}{2} & \text{for } \ x_1 \in [0,2), \ \ x_2 \in [0,e^{-1}) \\ \dfrac{(x_1 - \eta_2(x_2))(x_1-4)}{2} & \text{for } \ x_1 \in (2,4], \ \ x_2 \in [0,e^{-1}) \\ \dfrac{x_1(x_1-4)}{2} & \text{for } \ x_1 \in [0,4], \ \ x_2 \in (e^{-1},2] \end{cases}.$$

One can easily see that $u(x_1,x_2)$ is not a classical solution of problem (3), (4), since it is discontinuous along the line segment $0 \leq x_1 \leq 4$, $x_2 = e^{-1}$.

Remark 2. C^2 smoothness of the boundary of the domain $\mathbf{D}$ is very important and cannot be relaxed. Indeed, let $\alpha \in [1,2)$ be an arbitrary number,

$$\gamma_i(x_2) = 1 + (-1)^i \sqrt{1 - |x_2 - 1|^\alpha}(i = 1,2)$$

and

$$\eta_i(x_1) = 1 + (-1)^i x_1^{\frac{1}{\alpha}} (2 - x_1)^{\frac{1}{\alpha}} \ \ (i = 1,2).$$

In the domain

$$\begin{aligned}\mathbf{D} &= \Big\{(x_1,x_2):\ x_1 \in (0,2),\ x_2 \in \big(1 - x_1^{\frac{1}{\alpha}}(2-x_1)^{\frac{1}{\alpha}}, 1 + x_1^{\frac{1}{\alpha}}(2-x_1)^{\frac{1}{\alpha}}\big)\Big\}\\ &= \Big\{(x_1,x_2):\ x_2 \in (0,2),\ x_1 \in \big(1 - \sqrt{1-|x_2-1|^\alpha}, 1 + \sqrt{1-|x_2-1|^\alpha}\big)\Big\}\end{aligned}$$

consider the problem

$$u^{(2,2)} = 0, \tag{7}$$

$$u(\eta_i(x_2),x_2) = 0 \ \ (i = 1,2); \quad u^{(2,0)}(x_1,\gamma_i(x_1)) = 2 \ \ (i = 1,2). \tag{8}$$

It is clear that the only possible solution of problem $(7),(8)$ is a solution of the problem

$$u^{(2,0)} = 2, \tag{9}$$

$$u(\eta_i(x_2),x_2) = 0 \ \ (i = 1,2). \tag{10}$$

Problem $(9),(10)$ has the unique solution

$$\begin{aligned}u(x_1,x_2) &= \big(x_1 - 1 - \sqrt{1-|x_2-1|^\alpha}\,\big)\big(x_1 - 1 + \sqrt{1-|x_2-1|^\alpha}\,\big)\\ &= (x_1-1)^2 - 1 + |x_2-1|^\alpha = x_1^2 - 2x_1 + |x_2-1|^\alpha.\end{aligned}$$

However, $u^{(0,2)}(x_1,x_2)$ is discontinuous along the line segment $0 \le x_1 \le 2$, $x_2 = 1$, since $\alpha \in [1,2)$. Thus, problem $(7),(8)$ is not solvable in classical sense due to the fact that the boundary $\partial\mathbf{D}$ is not C^2 smooth at points $(0,1)$ and $(2,1)$.

Consider the quasilinear equation

$$\begin{aligned}u^{(2,2)} &= \rho_1\big(x_1,x_2,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\big)u^{(2,0)} + \rho_2\big(x_1,x_2,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\big)u^{(0,2)}\\ &+ \sum_{j=0}^{1}\sum_{k=0}^{1}\rho_{jk}\big(x_1,x_2,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\big)u^{(j,k)} + q\big(x_1,x_2,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}\big),\end{aligned} \tag{11}$$

where $\rho_i(x_1,x_2,\mathbf{z})$ $(i=1,2)$, $\rho_{jk}(x_1,x_2,\mathbf{z})$ $(j,k=0,1)$ and $q(x_1,x_2,\mathbf{z})$ are continuous functions on $\overline{\mathbf{D}} \times \mathbb{R}^4$, and $\mathbf{z} = (z_1,z_2,z_3,z_4)$.

Theorem 2. *Let $\rho_i \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$ $(i=1,2)$, $\rho_{jk} \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$ $(j,k=0,1)$, $q \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$, $\phi_i \in C^2([0,\omega_2])$, $\psi_i \in C^2([0,\omega_1])$ $(i=1,2)$, and let there exist functions $P_{il} \in C(\overline{\mathbf{D}})$ $(i,l=1,2)$ and $P_{ijk} \in C(\overline{\mathbf{D}})$ $(i,j=0,1;\ j,k=0,1)$ such that:*

(A_0)

$$0 \le P_{1l}(x_1,x_2) \le \rho_l(x,y,\mathbf{z}) \le P_{2l}(x_1,x_2) \ \ \textit{for} \ \ (x_1,x_2,\mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \quad (l=1,2);$$

(A_1)

$$P_{1jk}(x_1,x_2) \le \rho_{jk}(x_1,x_2,\mathbf{z}) \le \ P_{2jk}(x_1,x_2) \ \ \textit{for} \ \ (x_1,x_2,\mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \quad (j,k=0,1);$$

(A$_2$) *for arbitrary measurable functions* $p_i : \overline{\mathbf{D}} \to \mathbb{R}$ $(i = 1, 2)$ *and* $p_{jk} : \overline{\mathbf{D}} \to \mathbb{R}$ $(j, k = 0, 1)$ *satisfying the inequalities*

$$P_{1l}(x_1, x_2) \le p_l(x, y) \le P_{2l}(x_1, x_2) \ \text{ for } \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \ \ (l = 1, 2),$$
$$P_{1jk}(x_1, x_2) \le p_{jk}(x_1, x_2) \le \ P_{2jk}(x_1, x_2) \ \text{ for } \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \ \ (j, k = 0, 1),$$

problem $(1_0), (2_0)$ *has only the trivial solution;*

(A$_3$)

$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{q(x_1, x_2, \mathbf{z})}{\|\mathbf{z}\|} = 0 \quad uniformly \ on \ \overline{\mathbf{D}}.$$

Then problem $(11), (2)$ *has at least one solution.*

Consider the linear and quasilinear equations

$$u^{(2,2)} = \big(p_1(x_1, x_2) u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1, x_2) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2) u + q(x_1, x_2), \tag{12}$$

$$u^{(2,2)} = \big(p_1(x_1, x_2, u) u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1, x_2, u) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2, u) + q\big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}\big) \tag{13}$$

and

$$u^{(2,2)} = \big(p_1(x_1, x_2) u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1, x_2) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2, u) + q(x_1, x_2). \tag{14}$$

Theorem 3. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain with* C^2 *boundary inscribed in* Ω *such that*

$$\begin{aligned}\mathbf{D} &= \big\{(x_1, x_2) \in \Omega : \ x_1 \in (0, \omega_1), \ x_2 \in (\gamma_1(x_1), \gamma_2(x_1))\big\} \\ &= \big\{(x_1, x_2) \in \Omega : \ x_2 \in (0, \omega_2), \ x_1 \in (\eta_1(x_2), \eta_2(x_2))\big\},\end{aligned}$$

where $\gamma_i \in C([0, \omega_1]) \cap C^2((0, \omega_1))$, $\eta_i \in C([0, \omega_2]) \cap C^2((0, \omega_2))$ $(i = 1, 2)$,

$$(-1)^i \gamma_i''(x_1) \le 0 \ \text{ for } \ x_1 \in (0, \omega_1) \ \ (i = 1, 2),$$
$$(-1)^i \eta_i''(x_2) \le 0 \ \text{ for } \ x_2 \in (0, \omega_2) \ \ (i = 1, 2),$$

and

$$\gamma_1(\xi_1^*) = 0, \quad \gamma_2(\xi_2^*) = \omega_2, \quad \eta_1(\zeta_1^*) = 0, \quad \eta_2(\zeta_2^*) = \omega_1$$

for some $\xi_1^*, \xi_2^* \in [0, \omega_1]$ *and* $\zeta_1^*, \zeta_2^* \in [0, \omega_2]$. *Furthermore, let* $p_1 \in C^{1,0}(\overline{\Omega})$, $p_2 \in C^{0,1}(\overline{\Omega})$, $p_0, q \in C(\overline{\Omega})$, $\phi_i \in C^2([0, \omega_2])$, $\psi_i \in C^2([0, \omega_1])$ $(i = 1, 2)$, *and let*

$$p_1(x_1, x_2) \ge 0, \quad p_2(x_1, x_2) \ge 0, \quad p_0(x_1, x_2) \le 0 \ \text{ for } \ (x_1, x_2) \in \mathbf{D}.$$

Then problem $(12), (2)$ *is uniquely solvable, and its solution in a unique way can be continued to a solution of equation* (12) *in the domain* Ω.

Furthermore, if

$$(-1)^i \gamma_i''(x_1) < 0 \ \text{ for } \ x_1 \in (0, \omega_1) \ \ (i = 1, 2) \tag{15}$$

and

$$(-1)^i \eta_i''(x_2) < 0 \ \text{ for } \ x_2 \in (0, \omega_2) \ \ (i = 1, 2), \tag{16}$$

then the solution of problem $(12), (2)$ *can be continued to a solution of equation* (12) *in the closed domain* $\overline{\Omega}$.

Theorem 4. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain same as in Theorem 3, and let* $p_1 \in C^{1,0,1}(\overline{\mathbf{D}} \times \mathbb{R})$, $p_2 \in C^{0,1,1}(\overline{\mathbf{D}} \times \mathbb{R})$, $p_0 \in C(\overline{\mathbf{D}} \times \mathbb{R})$, $q \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$, *and a nonnegative number* M *be such that*

$$p_1(x_1,x_2,z) \geq 0, \quad p_2(x_1,x_2,z) \geq 0 \ \textit{ for } \ (x_1,x_2,z) \in \overline{\mathbf{D}} \times \mathbb{R},$$
$$p_0(x_1,x_2,z)z \leq M \ \textit{ for } \ (x_1,x_2,z) \in \overline{\mathbf{D}} \times \mathbb{R},$$
$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{q(x_1,x_2,\mathbf{z})}{\|\mathbf{z}\|} = 0 \ \textit{ uniformly on } \ \overline{\mathbf{D}}.$$

Then problem (13), (2) *has at least one solution. Moreover, if inequalities* (15) *and* (16) *hold, then every solution of problem* (13), (2) *belongs to* $C^{2,2}(\overline{\mathbf{D}})$.

Corollary 1. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain same as in Theorem 3, let* $p_1 \in C^{1,0}(\overline{\mathbf{D}})$, $p_2 \in C^{0,1}(\overline{\mathbf{D}})$, $p_0 \in C(\overline{\mathbf{D}} \times \mathbb{R})$, $q \in C(\overline{\mathbf{D}})$, *and let*

$$\big(p_0(x_1,x_2,z_1) - p_0(x_1,x_2,z_1)\big)(z_1 - z_2) \leq 0 \ \textit{ for } \ (x_1,x_2,z) \in \overline{\mathbf{D}} \times \mathbb{R}.$$

Then problem (14), (2) *has one and only one solution. Moreover, if inequalities* (15) *and* (16) *hold, then the solution of problem* (13), (2) *belongs to* $C^{2,2}(\overline{\mathbf{D}})$.

Remark 3. Under the conditions of Theorem 3 the functions p_0, p_1 and p_2 may have arbitrary growth order with respect to the phase variable. As an example, consider the equation

$$\begin{aligned} u^{(2,2)} &= \big(e^{\alpha_1(x_1,x_2)u^2} u^{(1,0)}\big)^{(1,0)} + \big(e^{\alpha_2(x_1,x_2)u^3} u^{(0,1)}\big)^{(0,1)} - u^{2n+1} \\ &\quad + \sum_{k=0}^{2n} \beta_k(x_1,x_2)u^k + \Big(1 + |u| + |u^{(1,0)}| + |u^{(0,1)}| + |u^{(1,1)}|\Big)^{1-\varepsilon}, \end{aligned} \tag{17}$$

where $\alpha_1 \in C^{1,0}(\overline{\mathbf{D}})$, $\alpha_2 \in C^{0,1}(\overline{\mathbf{D}})$, $\beta_k \in C(\overline{\mathbf{D}})$ $(k = 0, \dots, 2n)$ are arbitrary functions, n is an arbitrary positive integer, and $\varepsilon \in (0,1)$. By Theorem 4, problem (17), (2) has at least one solution.

References

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