Emden–Fowler Type Differential Equations Possessing Kurzweil's Property

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M. Jasný and J. Kurzweil [1, 2] was the first who revealed the fact that unlike the second order linear differential equations, the Emden-Fowler type nonlinear differential equation

$$u'' = p(t)|u|^{\lambda} \operatorname{sgn}(u),$$

where $\lambda = const > 1$, and $p : [a, +\infty) \to \,]-\infty, 0[$ is a continuous function, may have simultaneously oscillatory and nonoscillatory solutions.

According to F. V. Atkinson's theorem [3], from the proven by J. Kurzweil [2] oscillation theorem it follows that if the function $t \mapsto t^{\frac{\lambda+3}{2}} |p(t)|$ is nondecreasing and

$$\int\limits_a^{+\infty} t|p(t)|\,dt < +\infty,$$

then the above-mentioned Emden–Fowler type equation along with oscillatory solutions has also separated from zero slowly growing solutions. Such type of theorems for different classes of superlinear and sublinear differential equations of second and fourth order have been proven in [4–8].

We have established unimprovable in a certain sense conditions guaranteeing the fact that the higher order Emden–Fowler type differential equation

$$u^{(n)} = p(t)|u|^{\lambda(|u|)} \operatorname{sgn}(u) \tag{1}$$

has Kurzweil's property. Here, $n > 3$, $p : [a, +\infty[\to \mathbb{R}$ is a function, Lebesgue integrable on every finite interval contained in $[a, +\infty[$, $a > 0$, and $\lambda : [0, +\infty[\to \mathbb{R}$ is a continuous function. Moreover, the function p satisfies the inequality

$$(-1)^{n-n_0} p(t) \le 0 \ \text{ for } \ t \ge a, \tag{2}$$

where n_0 is the integer part of number $\frac{n}{2}$, and the function λ satisfies either the condition

$$1 < \lambda(x) \le \lambda(y) \ \text{ for } \ 0 < x < y < +\infty, \tag{3}$$

or the conditions

$$\begin{gathered} \lambda(0) > 1, \ \ \lambda(x) \ge \lambda(y) \ \text{ for } \ 0 \le x < y < +\infty, \ \ -\infty < \lambda_0 = \lim_{x \to +\infty} \lambda(x) < 1, \\ \limsup_{x \to +\infty} (\lambda(x) - \lambda_0) \ln(x) < +\infty. \end{gathered} \tag{4}$$

Let $t_0 \in [a, +\infty[$. The solution $u : [t_0, +\infty[\to \mathbb{R}$ of equation (1) is said to be **proper** if it is not identically equal to zero in non of the neighborhood of $+\infty$.

The proper solution $u : [t_0, +\infty[\to \mathbb{R}$ is called:

1) **oscillatory** if it changes its sign in any neighborhood of $+\infty$ and **nonoscillatory**, otherwise;

2) **Kneser solution** if
$$u(t) \neq 0, \ \ (-1)^i u^{(i)}(t)u(t) \geq 0 \ \text{ for } \ t \geq t_0 \ \ (i = 1, \dots, n-1);$$

3) **vanishing at infinity** if the equality
$$\lim_{t \to +\infty} u(t) = 0$$
is fulfilled, and **separated from zero** if the inequality
$$\liminf_{t \to +\infty} |u(t)| > 0$$
is fulfilled;

4) **slowly growing** if
$$\limsup_{t \to +\infty} |u^{(n-1)}(t)| < +\infty$$
and **rapidly growing** if
$$\lim_{t \to +\infty} |u^{(n-1)}(t)| = +\infty.$$

Definition 1. Equation (1) **has property** K if it has a continuum of proper oscillatory solutions and a continuum of separated from zero slowly growing solutions.

Definition 2. Equation (1) **has property** K_0 if it has a continuum of proper oscillatory solutions, a continuum of separated from zero slowly growing solutions and a continuum of vanishing at infinity Kneser solutions.

Theorem 1. *Let n_0 be odd and along with* (2) *and* (3), *the condition*
$$\int_a^{+\infty} t^{n-2+\lambda(tx)} |p(t)| \, dt = +\infty \ \ \textit{for} \ \ x > 0 \tag{5}$$
be fulfilled. Then equation (1) *has property K if and only if*
$$\int_a^{+\infty} t^{n-1} |p(t)| \, dt < +\infty. \tag{6}$$

Theorem 1′. *Let $n = 2n_0 + 1$ ($n = 2n_0$), n_0 be odd and conditions* (2), (3), (5) *and* (6) *be fulfilled. Then every nonoscillatory proper solution of equation* (1) *is separated from zero Kneser solution (either is separated from zero Kneser solution, or rapidly growing solution).*

Theorem 2. *Let n_0 be even (odd) and along with* (2) *and* (4) *the condition*
$$\int_a^{+\infty} t^{n-n_0+(n_0-1)\lambda_0} |p(t)| \, dt = +\infty \tag{7}$$
be fulfilled. Then equation (1) *has property K (property K_0) if and only if*
$$\int_a^{+\infty} t^{(n-1)\lambda_0} |p(t)| \, dt < +\infty. \tag{8}$$

Theorem 2′. *Let n_0 be even (odd) and conditions* (2), (4), (7) *and* (8) *be fulfilled. Then every proper nonoscillatory solution of equation* (1) *is separated from zero slowly growing (either is separated from zero slowly growing, or vanishing at infinity Kneser solution).*

Example. Let

$$\lambda(x) = \lambda_0 + \frac{\lambda_1}{1+|x|}\,, \quad \text{where} \ \ \lambda_0 \in \,]-\infty, 1[\,, \ \ \lambda_1 > 1 - \lambda_0. \tag{9}$$

Then conditions (4) are fulfilled. Therefore, if n_0 is even (is odd) and the function p satisfies conditions (2), (7) and (8), then equation (1) has property K (property K_0). Moreover, every proper nonoscillatory solution of that equation is separated from zero slowly growing (either is separated from zero slowly growing, or vanishing at infinity Kneser solution).

Remark. Condition (7) in Theorems 2 and 2′ is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_0^{+\infty} t^{n-n_0+(n_0-1)\lambda_0+\varepsilon} |p(t)|\, dt = +\infty,$$

no matter how small $\varepsilon > 0$ is.

Finally, it should be noted that in the case $n = 3$ the question on the validity of Theorems 1 and 2 remains open. In particular, the following problems remain unsolved.

Problem 1. *Let* $n = 3$, $\lambda(x) \equiv \lambda_0 > 1$,

$$p(t) \le 0 \ \ \textit{for} \ \ t \ge a, \qquad \int\limits_a^{+\infty} t^{1+\lambda_0} |p(t)|\, dt = +\infty, \qquad \int\limits_a^{+\infty} t^2 |p(t)|\, dt < +\infty.$$

Then, does equation (1) *have at least one proper oscillatory solution or not?*

Problem 2. *Let* $n = 3$ *and along with* (9) *the conditions*

$$p(t) \le 0 \ \ \textit{for} \ \ t \ge a, \qquad \int\limits_a^{+\infty} t^2 |p(t)|\, dt = +\infty, \qquad \int\limits_a^{+\infty} t^{2\lambda_0} |p(t)|\, dt < +\infty$$

be fulfilled. Then, does equation (1) *have at least one proper oscillatory solution or not?*

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