Existence and Multiplicity of Periodic Solutions to Second-Order Differential Equations with Attractive Singularities

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Consider a second-order ordinary differential equation of the form

$$u'' + \frac{g(t)}{u^\lambda} = h(t)u^\delta + \mu f(t), \tag{1}$$

where $g, h, f \in L(\mathbb{R}/T\mathbb{Z})$, $g(t) \geq 0$ for a.e. $t \in \mathbb{R}$, $\overline{g} > 0$, $\overline{h} < 0$, $\overline{f} > 0$, $\lambda > 0$, $\delta \in (0,1)$, and $\mu \geq 0$ is a parameter.

Throughout we use the following notation.

- $C(\mathbb{R}/T\mathbb{Z})$ is a Banach space of T-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}$ endowed with a norm $\|u\|_C = \max\{|u(t)| : t \in [0,T]\}$.
- $AC^1(\mathbb{R}/T\mathbb{Z})$ is a set of T-periodic functions $u : \mathbb{R} \to \mathbb{R}$ such that u and u' are absolutely continuous.
- $L^p(\mathbb{R}/T\mathbb{Z})$ $(p \geq 1)$ is a Banach space of T-periodic functions $h : \mathbb{R} \to \mathbb{R}$ that are integrable with the p-th power on the interval $[0,T]$ endowed with a norm

$$\|h\|_p = \bigg(\int_0^T |h(s)|^p \, ds\bigg)^{1/p}.$$

  We write $L(\mathbb{R}/T\mathbb{Z})$ instead of $L^1(\mathbb{R}/T\mathbb{Z})$.

- $[x]_+ = \frac{1}{2}\,(|x| + x)$, $[x]_- = \frac{1}{2}\,(|x| - x)$.
- If $h \in L(\mathbb{R}/T\mathbb{Z})$ then $\overline{h} = \frac{1}{T}\int_0^T h(s)\,ds$.

By a T-periodic solution to (1) we understand a function $u \in AC^1(\mathbb{R}/T\mathbb{Z})$ which is positive and satisfies the equality (1) for almost every $t \in \mathbb{R}$.

Theorem 1. *Let $[h]_+, [f]_+ \in L^p(\mathbb{R}/T\mathbb{Z})$ with $p \geq 1$. Let, moreover, there exist $\varphi \in L^q(\mathbb{R}/T\mathbb{Z})$ $(q \geq 1)$ such that*[1]

$$[h]_+(t) + [f]_+(t) \leq \varphi(t) g^{\frac{q-1}{q}}(t) \ \text{ for a.e. } t \in \mathbb{R}$$

and let

$$\lim_{x \to t_+} \int\limits_x^{t+T/2} \frac{g(s)}{(s-t)^{\frac{\lambda(2p-1)q}{p}}}\, ds + \lim_{x \to t_-} \int\limits_{t+T/2}^{x+T} \frac{g(s)}{(t+T-s)^{\frac{\lambda(2p-1)q}{p}}}\, ds = +\infty$$

be fulfilled for every $t \in \mathbb{R}$. Then there exist $\mu^* \geq \mu_* > 0$ such that

- *Eq. (1) has at least two T-periodic solutions provided $\mu > \mu^*$;*
- *Eq. (1) has at least one T-periodic solution provided $\mu = \mu^*$;*
- *Eq. (1) has no T-periodic solution provided $\mu \in [0, \mu_*)$.*

Remark. In the case when $h(t) \leq 0$ for a. e. $t \in \mathbb{R}$ it can be proved that the numbers μ^* and μ_* appearing in Theorem 1 coincide.

Before we pass to the proof of Theorem 1, we introduce some definitions and notation.

Definition 1. We say that $\alpha, \beta \in AC^1(\mathbb{R}/T\mathbb{Z})$ are, respectively, lower and upper functions to the T-periodic problem for (1), if they are positive and

$$\alpha''(t) + \frac{g(t)}{\alpha^\lambda(t)} \geq h(t)\alpha^\delta(t) + \mu f(t) \ \text{ for a.e. } t \in \mathbb{R},$$

resp.

$$\beta''(t) + \frac{g(t)}{\beta^\lambda(t)} \leq h(t)\beta^\delta(t) + \mu f(t) \ \text{ for a.e. } t \in \mathbb{R}.$$

Definition 2. We say that a lower function α and an upper function β to the T-periodic problem for (1) are well-ordered if

$$\alpha(t) \leq \beta(t) \ \text{ for } t \in \mathbb{R}.$$

Definition 3. We say that a lower function α, resp. an upper function β to the T-periodic problem for (1) is strict if the inequality

$$\alpha(t) \leq u(t), \ \text{ resp. } u(t) \leq \beta(t) \ \text{ for } t \in \mathbb{R}$$

implies

$$\alpha(t) < u(t), \ \text{ resp. } u(t) < \beta(t) \ \text{ for } t \in \mathbb{R}$$

provided u is a T-periodic solution to (1).

Notation. We will write $\alpha(t;\mu)$, $\beta(t;\mu)$, or $u(t;\mu)$ to emphasize that the lower function α, the upper function β, or the solution u to the T-periodic problem for (1) corresponds to the particular parameter μ.

Sketch of the proof of Theorem 1. First we show that every T-periodic solution u to (1) is bounded from above. In particular, the following assertion holds.

[1]If $q = 1$ then we put $g^{\frac{q-1}{q}}(t) = 1$ for $t \in \mathbb{R}$.

Lemma 1. *There exists a non-decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $\mu > 0$ we have*

$$u(t;\mu) < \rho(\mu)$$

provided u is a T-periodic solution to (1).

A condition $\delta > 0$ is essential in the proof of Lemma 1. The next step is a construction of well-ordered strict lower and upper functions to the T-periodic problem for (1).

Lemma 2. *Let the assumptions of Theorem 1 be fulfilled. Then for every $\mu > 0$ there exists a strict lower function α to the T-periodic problem for* (1). *Moreover,*

$$\alpha(t;\mu) < u(t;\mu) \ \text{ for } \ t \in \mathbb{R}, \ \ \mu > 0$$

whenever u is a T-periodic solution to (1).

An important property of the lower functions $\alpha(t;\mu)$ appearing in Lemma 2 is that they are constructed in such a way that

$$\alpha(t;\mu_1) \le \alpha(t;\mu_2) \ \text{ for } \ t \in \mathbb{R} \ \text{ whenever } \ \mu_1 \ge \mu_2.$$

Lemma 3. *For every μ sufficiently large there exists a strict upper function β to the T-periodic problem for* (1) *such that*

$$\alpha(t;\mu) < \beta(t;\mu) < \rho(\mu) \ \text{ for } \ t \in \mathbb{R},$$

where ρ, resp. α are functions appearing in Lemma 1, resp. Lemma 2.

Now the condition $\delta < 1$ is essential in construction of the upper functions β in Lemma 3.

The next step is obvious – for sufficiently large μ we have constructed well-ordered lower and upper functions α and β. Therefore there exists at least one T-periodic solution u to (1) between them. Moreover, since α and β are strict, we have

$$\alpha(t;\mu) < u(t;\mu) < \beta(t;\mu) \ \text{ for } \ t \in \mathbb{R}, \ \ \mu \text{ sufficiently large.}$$

Furthermore, if we rewrite T-periodic problem for (1) in an equivalent operator form

$$u = M_\mu[u]$$

then it follows that the Leray-Schauder degree of the operator $I - M_\mu$ over the set

$$\Omega_\mu \overset{def}{=} \big\{ x \in C(\mathbb{R}/T\mathbb{Z}) : \ \alpha(t;\mu) < x(t) < \beta(t;\mu) \text{ for } t \in \mathbb{R} \big\}$$

is different from zero. More precisley,

$$d_{LS}(I - M_\mu, \Omega_\mu, 0) = 1 \ \text{ for } \ \mu \text{ sufficiently large.} \tag{2}$$

Thus we have proved the existence of at least one T-periodic solution to (1) in Ω_μ (for every μ sufficiently large), and have established the relation (2).

On the other hand, the following assertion holds.

Lemma 4. *Let the assumptions of Theorem 1 be fulfilled. Then there exists $\mu_* > 0$ such that there is no T-periodic solution to* (1) *with $\mu \in [0, \mu_*)$.*

For every $\mu > 0$ we define a set

$$\Psi_\mu \overset{def}{=} \big\{x \in C(\mathbb{R}/T\mathbb{Z}) : \ \alpha(t;\mu) < x(t) < \rho(\mu) \text{ for } t \in \mathbb{R}\big\}.$$

Let μ_0 be arbitrary but fixed and let, moreover, it be sufficiently large such that

$$d_{LS}(I - M_{\mu_0}, \Omega_{\mu_0}, 0) = 1.$$

Then, according to Lemma 4 we have

$$d_{LS}(I - M_\mu, \Psi_{\mu_0}, 0) = 0 \ \text{ for } \ \mu \in [0, \mu_*).$$

Furthermore, due to the fact that ρ is non-decreasing and α is non-increasing with respect to μ, from Lemmas 1 and 2 it follows that there is no T-periodic solution to (1) on $\partial\Psi_{\mu_0}$ for $\mu \in [\mu_*, \mu_0]$. Consequently,

$$d_{LS}(I - M_{\mu_0}, \Psi_{\mu_0}, 0) = 0.$$

Now, in view of Lemma 3 we have $\Omega_{\mu_0} \subsetneq \Psi_{\mu_0}$, and so the additive property of the Leray-Schauder degree results in

$$d_{LS}(I - M_{\mu_0}, \Psi_{\mu_0} \setminus \Omega_{\mu_0}, 0) = -1,$$

i.e., there is another T-periodic solution to (1) in $\Psi_{\mu_0} \setminus \Omega_{\mu_0}$.

Now define

$$A \overset{def}{=} \big\{\tau > 0 : \ \text{Eq. (1) has at least two } T\text{-periodic solutions for every } \mu \geq \tau\big\}.$$

Obviously, on account of the above-proven, the set A is nonempty. Moreover, according to Lemma 4, the set A is bounded from below by μ_*. Put

$$\mu^* \overset{def}{=} \inf A,$$

and let $\{\mu_n\}_{n=1}^{+\infty}$ be a sequence of parameters such that

$$\mu_n > \mu^* \ \text{ and } \ \lim_{n \to +\infty} \mu_n = \mu^*.$$

Obviously, there exist a sequence of T-periodic solutions $\{u(\,\cdot\,;\mu_n)\}_{n=1}^{+\infty}$ to (1) (with $\mu = \mu_n$). In addition, with respect to Lemmas 1 and 2, this sequence of solutions is uniformly bounded and equicontinuous. Thus, by standard arguments one can prove that there exists also at least one T-periodic solution to (1) with $\mu = \mu^*$. Now the sketch of the proof of Theorem 1 is complete.