Asymptotic of Rapid Varying Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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We consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ and } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

It means that in the considered equation the continuous function φ and its first order derivative are [6, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92] rapidly varying as $y \to Y_0$.

For two-term differential equations of second order with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works by M. Marić [6], V. M. Evtukhov and his students N. G. Drik, A. G. Chernikova [2,3].

In the works by V. M. Evtukhov, A. G. Chernikova [3] for the differential equation (1) of second order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0, \lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

In the works by V. M. Evtukhov, N. V. Sharay [5] for the differential equation (1) of third order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0, \lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R} \setminus \{0, 1, \frac{1}{2}\}$. In this work, we propose the distribution of [3] results to third-order differential equations.

Solution y of the differential equation (1) specified on the interval $[t_0, \omega[\subset [a, \omega[$ is said to be $P_\omega(Y_0, \lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad k = 1, 2, \quad \lim_{t \uparrow \omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1) in the non-singular case when $\lambda_0\in\mathbb{R}\setminus\{0,1,\frac12\}$, as well as asymptotic representations as $t\uparrow\omega$ for such solutions and their derivatives up to the second order inclusively.

Without loss of generality, we assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is the left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is the right neighborhood of } Y_0,\end{cases} \tag{3}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$, when $Y_0=0$ and $y_0>1$ $(y_0<-1)$, when $Y_0=+\infty$ (when $Y_0=-\infty$).

A function $\varphi:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), belongs to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the work [3] which extends the class of function Γ, introduced by L. Khan (see, for example, [1, Ch. 3, § 3.10, p. 175]). Using properties from this class the main results are obtained.

We introduce the necessary auxiliary notation. We assume that the domain of the function $\varphi\in\Gamma_{Y_0}(Z_0)$ is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$\pi_\omega(t)=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\qquad J_1(t)=\int\limits_{A_1}^{t}p^{\frac13}(\tau)\,d\tau,\quad \Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)},$$

where

$$A_1=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=const,\\ a & \text{if } \displaystyle\int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=\pm\infty,\end{cases}\qquad B_1=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=const,\\ y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=\pm\infty.\end{cases}$$

Considering the definition of $P_\omega(Y_0,1)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,1)$-solution and of its first derivative in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0,\ \text{if } Y_0=0,\quad \nu_0\nu_1>0,\ \text{if } Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$ as $y\to Y_0$ and increases by Δ_{Y_0}, because on this interval $\Phi_1'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi_1^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ_1^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y)=\begin{cases}\text{eitherr} & 0,\\ \text{or} & +\infty,\end{cases}\qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{or } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{or } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\Phi_1(y_0).$$

In addition to the indicated notation, using Φ_1^{-1} we also introduce the auxiliary functions

$$q_1(t) = \frac{\alpha_0 \nu_1 J_1(t)}{p^{\frac{1}{3}}(t)\varphi^{\frac{1}{3}}(\Phi_1^{-1}(\nu_1 J_1(t)))(\Phi_1^{-1}(\nu_1 J_1(t)))^{\frac{2}{3}}},$$

$$H_1(t) = \frac{\Phi_1^{-1}(\nu_1 J_1(t))\varphi'(\Phi_1^{-1}(\nu_1 J_1(t)))}{\varphi(\Phi_1^{-1}(\nu_1 J_1(t)))},$$

$$J_2(t) = \int_{A_2}^{t} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau, \quad J_3(t) = \int_{A_3}^{t} J_2(\tau)\,d\tau,$$

where

$$A_2 = \begin{cases} t_0 & \text{if } \int\limits_{t_2}^{\omega} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{a}^{\omega} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau < +\infty, \end{cases}$$

$$A_3 = \begin{cases} t_0 & \text{if } \int\limits_{t_3}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{a}^{\omega} J_2(\tau)\,d\tau < +\infty, \end{cases} \qquad t_2, t_3 \in [a, \omega).$$

For equation (1) the following assertions are valid.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$. For the existence of $P_\omega(Y_0, 1)$-solutions of the differential equation* (1)*, it is necessary to comply with the conditions*

$$\alpha_0 \nu_0 > 0, \quad \mu_0 \nu_1 J_1(t) > 0 \mathit{for} \ \ t \in (a, \omega);$$

$$\nu_1 \lim_{t \uparrow \omega} J_1(t) = Z_0, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_1'(t)}{J_1(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q(t) = 1$$

and

$$\lim_{t \uparrow \omega} \frac{p(t)\varphi\big(\Phi_1^{-1}(\nu_1 J_1(t))\big) J_3(t)}{(J_2(t))^2} = 1.$$

Moreover, for each such solution there take place the asymptotic representations

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0 - 1) J_1(t)\big)\Big[1 + \frac{o(1)}{H_1(t)}\Big] \ \ \mathit{as} \ \ t \uparrow \omega,$$

$$y'(t) = \nu_1 p^{\frac{1}{3}}(t)\varphi^{\frac{1}{3}}\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)^{\frac{2}{3}}[1 + o(1)] \ \ \mathit{as} \ \ t \uparrow \omega,$$

$$y''(t) = \alpha_0 J_2(t)[1 + o(1)] \ \ \mathit{as} \ \ t \uparrow \omega.$$

In addition, sufficient conditions for the existence of such solutions are obtained.

References

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