Optimization of the Delay Parameter for One Class of Controlled Dynamical System

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1 Mathematical model

As is known the real controlled dynamical systems contain effects with delayed action and are described by differential equations with delay in control [3]. To illustrate this, below we will consider the simplest model of marketing relation.

Let $t_1 > t_0$, $\beta > \alpha \geq 0$ and $\theta_2 > \theta_1 > 0$ be given numbers. Let market relation demand and supply be described by the functions $D(t,p)$ and $S(t,q)$, which are continuous and continuously differentiable with respect to p and q.

Let the function $p(t) \in P = [\alpha, \beta]$, $t \in I_1 = [t_0 - \theta_2, t_1]$ be price of a good, changing over time. Suppose that at time $t \in I_2 = [t_0, t_1]$ will be satisfied demand of consumer which has been ordered at time $t - \theta$, i.e. when price of a good was $p(t - \theta)$. Here $\theta \in I_3 = [\theta_1, \theta_2]$ is so-called delay parameter.

The function

$$R(t) = D(t, p(t)) - S(t, p(t - \theta)), \ \ t \in I_2,$$

we call the disbalance index.

If $R(t) = 0$, then at the moment t we do not have disbalance between supply and demand, and the customer will buy exactly the quantity of goods he needs.

It is clear that at various time moment t the disbalance index $R(t)$ is possible to be not positive as well as positive. At time t, if $R(t) > 0$, then demand exaggerates supply. If $R(t) < 0$, then supply exaggerates demand. To describe development of marketing relation process in time, i.e. create dynamical model, we consider the integral index of disbalance

$$y(t) = R(t_0) + \int_{t_0}^{t} R(s)\,ds. \tag{1.1}$$

The function $y(t)$ gives complete information about the disbalance from the initial time t_0 to any time t.

From (1.1) we get the differential equation

$$\dot{y}(t) = D(t, p(t)) - S(t, p(t - \theta)), \ \ t \in I_2 \tag{1.2}$$

with the initial condition

$$y(t_0) = y_0 := R(t_0).$$

2 Statement of the problem. Necessary optimality conditions

Let $O \subset \mathbb{R}^n$ be an open set and $U \subset \mathbb{R}^r$ be a convex and compact set. Let the $(n+1)$-dimensional function

$$F(t,x,u,v) = \big(f^0(t,x,u,v), f(t,x,u,v)\big)^\top,$$

where $f = (f^1,\dots,f^n)^\top$, be continuous on $I_2 \times O \times U^2$ and continuously differentiable with respect to x and u, v. Furthermore, let $x_0, x_1 \in O$ be fixed points and let Ω be a set of absolutely continuous control functions $u(t) \in U$, $t \in I_1$. To each element $w = (\theta, u) \in \Lambda := I_3 \times \Omega$ we assign the differential equation

$$\dot{x}(t) = f\big(t, x(t), u(t), u(t-\theta)\big), \;\; t \in (t_0, t_1) \tag{2.1}$$

with the initial condition

$$x(t_0) = x_0. \tag{2.2}$$

Definition 2.1. Let $w = (\theta, u) \in \Lambda$. A function $x(t) = x(t; w) \in O$, $t \in I_2$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element w and defined on the interval I_2 if it satisfies condition (2.2) and is continuously differentiable and satisfies equation (2.1) everywhere on (t_0, t_1).

Definition 2.2. An element $w = (\theta, u) \in \Lambda$ is said to be admissible if the corresponding solution $x(t) = x(t; w)$ satisfies the condition

$$x(t_1) = x_1. \tag{2.3}$$

Denote by Λ_0 the set of admissible elements.

Definition 2.3. An element $w_0 = (\theta_0, u_0) \in \Lambda_0$ is said to be optimal if for an arbitrary element $w \in \Lambda_0$ we have

$$J(w_0) \le J(w), \tag{2.4}$$

where

$$J(w) = \int_{t_0}^{t_1} f^0(t, x(t), u(t), u(t-\theta))\, dt$$

and $x(t) = x(t; w)$.

(2.1)–(2.4) is called the optimization problem of delay parameter θ and control $u(t)$.

Theorem 2.1. *Let w_0 be an optimal element and let $x_0(t) = x(t; w_0)$ be the optimal trajectory. Then there exists a nontrivial solution $\Psi(t) = (\psi_0(t), \psi(t))$ of the equation*

$$\dot{\psi}(t) = -\Psi(t) F_x[t], \tag{2.5}$$

where

$$F_x[t] = F_x\big(t, x_0(t), u_0(t), u_0(t-\theta_0)\big),$$

such that $\psi_0(t) \equiv const \le 0$ and the following conditions hold:

(i_1) *the integral condition for the optimal delay parameter θ_0*

$$\int_{t_0}^{t_1} \Psi(t) F_v[t] \dot{u}_0(t-\theta_0)\, dt = 0;$$

(i_2) *the integral maximum principle for the optimal control* $u_0(t)$

$$\int\limits_{t_0}^{t_1} \Psi(t)\big[F_u[t]u_0(t) + F_v[t]u_0(t-\theta_0)\big]\,dt = \max_{u(t)\in\Omega} \int\limits_{t_0}^{t_1} \Psi(t)\big[F_u[t]u(t) + F_v[t]u(t-\theta_0)\big]\,dt.$$

The necessary optimality condition for the delay parameter in controls for the optimization problem with the Meyer type functional is provided in [2].

3 Optimization problem for equation (1.2). Necessary optimality conditions

Let y_1 be a fixed number and let V be a set of absolutely continuous control functions $p(t) \in P$, $t \in I_1$. To each element $\vartheta = (\theta, p) \in \Pi := I_3 \times V$ we assign the differential equation

$$\dot{y} = D(t, p(t)) - S(t, p(t-\theta)), \ \ t \in I_2$$

with the initial condition

$$y(t_0) = y_0.$$

Definition 3.1. An element $\vartheta = (\theta, p) \in \Pi$ is said to be admissible if the corresponding solution $y(t) = y(t;\vartheta)$ satisfies the condition

$$y(t_1) = y_1.$$

Denote by Π_0 the set of admissible elements.

Definition 3.2. An element $\vartheta_0 = (\theta_0, p_0) \in \Pi_0$ is said to be optimal if for an arbitrary element $\vartheta \in \Pi_0$ we have

$$\int\limits_{t_0}^{t_1} g(t, p_0(t))\,dt \le \int\limits_{t_0}^{t_1} g(t, p(t))\,dt,$$

where the function $g(t,p)$ is continuous and continuously differentiable with respect to p.

It is clear that for the considered problem we have $\dot{\psi} = 0$ (see (2.5)). Taking into account the last equation from Theorem 2.1 it follows

Theorem 3.1. *Let* ϑ_0 *be an optimal element. Then there exists a nontrivial vector* $\Psi = (\psi_0, \psi)$, $\psi_0 \le 0$ *such that the following conditions hold:*

(i_3) *the integral condition for the optimal delay parameter* θ_0

$$\psi \int\limits_{t_0}^{t_1} S_q(t, p_0(t-\theta_0))\dot{p}_0(t-\theta_0)\,dt = 0;$$

(i_4) *the integral maximum principle for the optimal control* $p_0(t)$

$$\int\limits_{t_0}^{t_1} \Big[\big(\psi_0 g_p(t, p_0(t)) + \psi D_p(t, p_0(t))\big)p_0(t) - \psi S_q(t, p_0(t-\theta_0))p_0(t-\theta_0)\Big]\,dt$$

$$\max_{p(t)\in V} \int\limits_{t_0}^{t_1} \Big[\big(\psi_0 g_p(t, p_0(t)) + \psi D_p(t, p_0(t))\big)p(t) - \psi S_q(t, p_0(t-\theta_0))p(t-\theta_0)\Big]\,dt.$$

Analogous problem for equation (1.2) with the fixed θ is investigated in [1].

References

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