Asymptotic Representations of Rapid Varying Solutions of Differential Equations Asymptotically Close to the Equations with Regularly Varying Nonlinearities

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The differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \tag{1}$$

is considered. Here $n \geq 2$, $f : [a, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \times \dots \times \Delta_{Y_{n-1}} \to \mathbb{R}$ is some continuous function, $-\infty < a < \omega \leq +\infty$, Y_j equals to zero, or to $\pm\infty$, Δ_{Y_j} is some one-sided neighborhood of Y_j, $j = 0, 1, \dots, n-1$.

The asymptotic estimations for singular, quickly varying, and Kneser solutions of equation (1) are described in the monograph by I. T. Kiguradze, T. A. Chanturia [4].

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\, \subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if the next conditions take place

$$y^{(j)}(t) \in \Delta_{Y_j} \ \text{ as } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \ \ (j = 0, 1, \dots, n-1),$$
$$\lim_{t \uparrow \omega} \frac{\big[y^{(n-1)}(t)\big]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0.$$

The asymptotic behavior of such solutions earlier has been investigated in the works by V. M. Evtukhov and A. M. Klopot [1–3, 5] for the differential equation

$$y^{(n)} = \sum_{i=1}^{m} \alpha_i p_i(t) \prod_{j=0}^{n-1} \varphi_{ij}(y^{(j)}),$$

where $n \geq 2$, $\alpha_i \in \{-1; 1\}$, $p_i : [a, \omega[\, \to]0, +\infty[$ is a continuous function, $i = 1, \dots, m$, $-\infty < a < \omega \leq +\infty$, $\varphi_{ij} : \Delta_{Y_j} \to]0, +\infty[$ is a continuous regularly varying as $y^{(j)} \to Y_j$ function of order σ_j, $j = 0, 1, \dots, n-1$ $(i = 1, \dots, m)$.

The aim of the paper is in establishing the necessary and sufficient conditions of the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solutions of equation (1) and in finding the asymptotic representations of such solutions and their derivatives to the order $n-1$ including.

Every $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solution of the differential equation (1) has (see, for example, [1]) the next a priori asymptotic properties

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \dots \sim \frac{y^{(n)}(t)}{y^{(n-1)}(t)} \ \text{ as } \ t \uparrow \omega, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = \pm\infty,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega < +\infty. \end{cases}$$

Definition 2. The function f in the differential equation (1) is called a function, that satisfies the condition $(RN)_1$, if there exist a number $\alpha_0 \in \{-1,1\}$, a continuous function $p : [a,\omega[\to]0,+\infty[$ and continuous regularly varying as $z \to Y_j$ $(j=\overline{0,n-1})$ functions $\varphi_j : \Delta_{Y_j} \to]0,+\infty[$ $(j=\overline{0,n-1})$ of orders σ_j $(j=\overline{0,n-1})$, such that for all continuously differentiable functions $z_j : [a,\omega[\to \Delta_{Y_j}$ $(j=\overline{0,n-1})$, satisfying the conditions

$$\lim_{t\uparrow\omega} z_j(t)=Y_j,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)z_j'(t)}{z_j(t)}=\pm\infty \ \ (j=\overline{0,n-1}),$$
$$\lim_{t\uparrow\omega}\frac{z_{j-1}'(t)z_j(t)}{z_{j-1}(t)z_j'(t)}=1 \ \ (j=\overline{1,n-1}),$$

the next representation takes place

$$f(t,z_0(t),z_1(t),\dots,z_{n-1}(t))=\alpha_0 p(t)\prod_{j=0}^{n-1}\varphi_j(z_j(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Furthermore, we will use the following notations.

$$\gamma=1-\sum_{j=0}^{n-1}\sigma_j,\quad \mu_n=\sum_{j=0}^{n-2}\sigma_j(n-j-1);$$

$$\nu_j=\begin{cases}1 & \text{if } Y_j=+\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is the right neigbourhood of zero},\\ -1 & \text{if } Y_j=-\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is the left neigbourhood of zero}\end{cases}\quad (j=\overline{0,n-1});$$

$$J_0(t)=\int_{A_0}^t p(s)\,ds,\quad J_{00}(t)=\int_{A_{00}}^t J_0(s)\,ds,$$

where

$$A_0=\begin{cases}a & \text{if } \displaystyle\int_a^\omega p(s)\,ds=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega p(s)\,ds<+\infty,\end{cases}\qquad A_{00}=\begin{cases}a & \text{if } \displaystyle\int_a^\omega |J_0(s)|\,ds=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega |J_0(s)|\,ds<+\infty.\end{cases}$$

Theorem 1. *Let the function f satisfy the condition $(RN)_1$ and $\gamma\neq 0$. Then for the existence of $P_\omega(Y_0,\dots,Y_{n-1},1)$-solutions of equation (1) the next conditions are necessary:*

$$\frac{p(t)}{J_0(t)}\sim\frac{J_0(t)}{J_{00}(t)} \ \ as \ \ t\uparrow\omega,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{J_0(t)}=\pm\infty,$$
$$\nu_j\lim_{t\uparrow\omega}|J_0(t)|^{\frac{1}{\gamma}}=Y_j \ \ (j=\overline{0,n-1}),$$

and, for $t\in]a,\omega[$, the next inequalities take place

$$\alpha_0\nu_{n-1}\gamma J_0(t)>0,\quad \nu_j\nu_{n-1}(\gamma J_0(t))^{n-j-1}>0 \ \ (j=\overline{0,n-2}).$$

As the algebraic of ρ equation

$$(1+\rho)^n=\sum_{j=0}^{n-1}\sigma_j(1+\rho)^j \tag{2}$$

has no roots with zero real part, the conditions also are sufficient for the existence of such solutions of equation (1). *Moreover, for any such solution the next asymptotic representations*

$$y^{(j)}(t)=\Big(\frac{\gamma J_{00}(t)}{J_0(t)}\Big)^{n-j-1}y^{(n-1)}(t)[1+o(1)]\ \ (j=\overline{0,n-2}),\tag{3}$$

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=0}^{n-1}L_j\big((\frac{\gamma J_{00}(t)}{J_0(t)})^{n-j-1}y^{(n-1)}(t)\big)}=\alpha_0\nu_{n-1}\gamma J_0(t)\Big|\frac{\gamma J_{00}(t)}{J_0(t)}\Big|^{\mu_n}[1+o(1)],\tag{4}$$

take place as $t\uparrow\omega$. Here $L_j(y^{(j)})=|y^{(j)}|^{-\sigma_j}\varphi_j(y^{(j)}t)$ $(j=\overline{0,n-1})$. There exists m-parametric family of such solutions, if among the roots of equation (2) *there exist m roots (taking into account multiply roots), the real parts of which have the sign that is among opposite to the sign $\alpha_0\nu_{n-1}$.*

The asymptotic representation of the $(n-1)$-th derivative of $P_\omega(Y_0,\dots,Y_{n-1},1)$-solution of equation (1) is given in the implicit form. We will indicate the conditions by implementation of which the asymptotic representations (3), (4) can be written in the explicit form.

Definition 3. The slowly varying as $y\to Y$ function $L:\Delta_Y\to]0,+\infty[$, where Y equals either zero, or $\pm\infty$, Δ_Y is a one-sided neighborhood of Y, is called satisfying the condition S_0 if the next condition takes place:

$$L(\nu e^{[1+o(1)]\ln|y|})=L(y)[1+o(1)]\ \text{ as }\ y\to Y\ \ (y\in\Delta_Y),$$

where $\nu=\operatorname{sign}y$.

Theorem 2. *Let the conditions of Theorem 1 be satisfied and regularly varying functions L_j $(j=\overline{0,n-1})$ satisfy the condition S_0. Then for any $P_\omega(Y_0,\dots,Y_{n-1},1)$-solution of equation* (1) *the next asymptotic representations*

$$y^{(j)}(t)=\nu_{n-1}\Big(\frac{\gamma J_0(t)}{p(t)}\Big)^{n-j-1}\bigg|\gamma J_0(t)\Big|\frac{\gamma J_0(t)}{p(t)}\Big|^{\mu_n}\prod_{j=0}^{n-1}L_j\big(\nu_j|J_0(t)|^{\frac{1}{\gamma}}\big)\bigg|^{\frac{1}{\gamma}}[1+o(1)]\ \ (j=\overline{0,n-1})$$

take place as $t\uparrow\omega$.

References

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