Decaying Solutions of Delay Differential Equations

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1 Introduction

Consider the differential equation with damping term

$$x'' = h(t, x(t), x(\gamma(t)))x'(t) + f(t, x(\tau(t)), x(t)), \tag{1.1}$$

where:

1. the functions γ, τ are continuous functions on $[t_0, \infty)$ such that $\gamma(t) \geq t_0, \tau(t) \geq t_0$ and $\lim\limits_{t\to\infty} \gamma(t) = \lim\limits_{t\to\infty} \tau(t) = \infty$;
2. the function h is a continuous function on $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$;
3. the function f is a continuous function on $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$ and

   $$0 < f(t, u, v) \leq b(t) \ \text{ for any } \ (u, v) \in (0, 1] \times (0, 1], \tag{Hp1}$$

   where b is a positive continuous function on $[t_0, \infty)$.

Let x be a solution of (1.1) and denote by H_x the function

$$H_x(t) = \exp\bigg(-\int\limits_{t_0}^{t} h\big(r, x(r), x(\gamma(r))\big)\, dr \bigg).$$

Hence (1.1) is equivalent to the functional equation

$$(H_x(t)x'(t)) = H_x(t) f\big(t, x(\tau(t)), x(t)\big), \tag{1.2}$$

that is an equation in which the differential operator depends also on the state x. Equations of this type model reaction-diffusion problems with non-constant diffusivity, see, e.g., the papers [6, 14] and the references therein.

A prototype of (1.2) is the nonlinear equation

$$(p(t)g(x)x'(t))' = f(t, x(\tau(t)), \tag{1.3}$$

where p is a positive continuously differentiable function on $[t_0, \infty)$ and g is a continuously positive differentiable function on $\mathbb{R}$. Equation (1.3) includes the well-known Thomas–Fermi equation, as well as the Schroedinger–Persico equation, which occur in the study of atomic fields, see [17]. Moreover, (1.3) arises also in some mechanical problems as the law of angular momentum conservation when the field strength is time dependent, see [11].

Our aim here is to present some results concerning solutions x of (1.1) satisfying

$$x(t) > 0, \quad x'(t) < 0 \ \text{ for large } t. \tag{1.4}$$

Further, the asymptotic behavior is also examined, jointly with some comments and open problems. These results are taken from [7] and are here presented without proofs.

Solutions of (1.1) satisfying (1.4) are usually called *Kneser solutions*. The Kneser existence problem and the asymptotic decay of Kneser solutions have been deeply studied in the case without deviating arguments. We refer the reader to the monograph [9], the papers [1,3] and the references therein. In the general case of equations with deviating arguments, we refer to the books [8, 12], the papers [10, 16] and the references therein.

2 Main results

Since $\lim\limits_{t\to\infty} \gamma(t) = \lim\limits_{t\to\infty} \tau(t) = \infty$, there exists $\widetilde{t} \geq t_0$ such that $\gamma(t) \geq t_0$ and $\tau(t) \geq t_0$ for any $t \in [\widetilde{t}, \infty)$. Thus, define t_1 such that

$$t_1 = \inf \big\{ \widetilde{t} \geq t_0 : \ \min\{\gamma(t), \tau(t)\} \geq t_0 \text{ on } [\widetilde{t}, \infty) \big\}. \tag{2.1}$$

Our main result is the following.

Theorem 2.1. *Assume that there exist two functions $\lambda \in C^1(I, \mathbb{R}^+)$, $\theta \in C(I, \mathbb{R})$ such that:*

(i_1) *for any $t \geq t_0$ and $0 \leq u \leq 1$, $0 \leq v \leq 1$*

$$h(t, u, v) \geq \frac{\lambda'(t)}{\lambda(t)}\,. \tag{2.2}$$

(i_2)

$$\int\limits_{t_1}^{\infty} \lambda(s) \int\limits_{s}^{\infty} \theta(r) b(r)\, \mathrm{d}r\, \mathrm{d}s < \infty. \tag{2.3}$$

(i_3) *for every $t > t_0$*

$$\lambda(t)\theta(t) > t_0. \tag{2.4}$$

Then the BVP (1.1), (1.4) *has at least one solution.*

Theorem 2.1 requires the existence of two auxiliary functions, namely λ and θ, satisfying certain properties. The following results give examples of such functions.

Corollary 2.1. *Assume that for some* $n \in \mathbb{N} \cup \{0\}$,

$$\int_{t_0}^{\infty} s^{-n} \int_{s}^{\infty} r^n b(r)\,\mathrm{d}r\,\mathrm{d}s < \infty.$$

If for any $t \geq t_0$ *and* $0 \leq u \leq 1$, $0 \leq v \leq 1$,

$$h(t,u,v) \geq -\frac{n}{t}\,, \tag{2.5}$$

then the BVP (1.1), (1.4) *has at least one solution.*

Proof. The assertion follows from Theorem 2.1 by choosing

$$\lambda(t) = t^{-n}, \quad \theta(t) = (t_0+1)t^n. \qquad \square$$

Corollary 2.2. *Assume that for some* $M > 0$,

$$\int_{t_0}^{\infty} \mathrm{e}^{-Ms} \int_{s}^{\infty} \mathrm{e}^{Mr} b(r)\,\mathrm{d}r\,\mathrm{d}s < \infty.$$

If for any $t \geq t_0$ *and* $0 \leq u \leq 1$, $0 \leq v \leq 1$,

$$h(t,u,v) \geq -M, \tag{2.6}$$

then BVP (1.1), (1.4) *has at least one solution.*

Proof. The assertion follows from Theorem 2.1 by choosing

$$\lambda(t) = \mathrm{e}^{-Mt}, \quad \theta(t) = (t_0+1)\mathrm{e}^{Mt}. \qquad \square$$

Remark 2.1. Observe that the assumption (2.5) in Corollary 2.1 and the assumption (2.6) in Corollary 2.2 permit us to choose damping terms which take negative values.

Remark 2.2. Theorem 2.1 does not require superlinear conditions (or sublinear conditions) on the forcing term f. Hence, it may be applicable in a wide variety of cases.

Remark 2.3. The proof of Theorem 2.1 is based on a fixed point theorem for multivalued operators which arises from [4]. The main advantage of this approach is that the explicit form of the fixed point operator is not needed, because the topological properties, like the compactness and continuity of the fixed point operator are obtained directly from the *a-priori* bounds for solutions of a suitable associated BVP.

In the sequel, consider the special case of (1.1)

$$x''(t) = h(t, x(t), x(\gamma(t)))x'(t) + \psi(t, x(\tau(t))), \tag{2.7}$$

where the functions γ, τ and h are as in (1.1), τ is a delay and ψ is a continuous function on $[t_0,\infty) \times \mathbb{R}$ such that

$$0 < \psi(t,u) \leq b(t) \ \text{ for any } \ u \in (0,1]. \tag{2.8}$$

Observe that in (2.7) the forcing term ψ depends on state x at time $\tau(t)$, but does not depend on x at time t.

Theorem 2.2. *Assume that:*

(i$_1$)
$$\int\limits_{t_0}^{\infty} t\, b(t)\,\mathrm{d}t < \infty. \tag{2.9}$$

(i$_2$) $\tau(t) < t$.

(i$_3$) *The function h is nonnegative on $[t_0,\infty)\times[0,1]\times[0,1]$.*

Then the equation (2.7) *has Kneser solutions x which satisfy*
$$x(t)x'(t) < 0 \ \ \text{on } t\in[t_1,\infty), \tag{2.10}$$
where t_1 is given in (2.1) *and*
$$\lim_{t\to\infty} x(t) = 0. \tag{2.11}$$

Theorem 2.2 shows a discrepancy between equations with or without delay, which is illustrated by the following example.

Example 2.1. Consider the equation
$$x''(t) = g(t)\sqrt{x^2(t)+x^2(\gamma(t))}\ x'(t) + \mathrm{e}^{-t}x(t-\pi), \tag{2.12}$$
where g is a nonnegative continuous function on $[t_0,\infty)$. In view of Theorem 2.2, equation (2.12) has Kneser solutions which satisfy (2.10) and (2.11). Observe that when $g\equiv 0$ on $[t_0,\infty)$, then any Kneser solution of the corresponding linear equation without delay
$$x''(t) = e^{-t}x(t)$$
does not converge to zero as $t\to\infty$, see, e.g. [15, Section 4].

3 Open problems

Open problem 1. Consider the Emden–Fowler equation
$$x''(t) = b(t)|x(t)|^{\beta}\operatorname{sgn} x(t) \tag{3.1}$$
and the corresponding equation with deviating argument
$$x''(t) = b(t)|x(\tau(t))|^{\beta}\operatorname{sgn} x(\tau(t)), \tag{3.2}$$
where b is a positive function on $[t_0,\infty)$ and $\tau(t) < t$.

First observe that if $\beta > 1$ and b is positive, then equation (3.1) always has Kneser solutions. Moreover, if in addition
$$\int\limits_{t_0}^{\infty} sb(s)\,\mathrm{d}s < \infty,$$
then (3.1) does not have Kneser solutions which tend to zero as $t\to\infty$, see, e.g., [3]. If $\tau(t) < t$, then this result may fail for (3.2) as Theorem 2.2 illustrates.

In the sublinear case, that is $0 < \beta < 1$, there might exist equations of type (3.1) without Kneser solutions. For instance, if

$$\liminf_{t\to\infty} t^2 b(t) > 0, \tag{3.3}$$

then (3.1) does not have Kneser solution, see [9, Corollary 17.3]. On the other hand, from Corollary 2.1 with $n = 0$ we get that the equation

$$x''(t) = \frac{1}{t^2 \log t}\, |x(\tau(t))|^{\beta} \operatorname{sgn} x(\tau(t)), \quad t \ge 2, \tag{3.4}$$

has Kneser solutions. For (3.4) we have

$$\liminf_{t\to\infty} t^2 b(t) = \liminf_{t\to\infty} \frac{1}{\log t} = 0.$$

Thus, it is an open problem if the Kiguradze condition (3.3) is sufficient for the nonexistence of Kneser solutions of (3.2) when $0 < \beta < 1$ and $\tau(t) - t \not\equiv 0$. Finally, observe that if

$$\tau(t) < t \ \text{ and } \ 0 < \beta < 1,$$

then, in view of Theorem 2.2, equation (3.4) has Kneser solutions which tend to zero as $t \to \infty$.

Open problem 2. Kneser solutions which are decaying to zero as $t \to \infty$ may have a different asymptotic behavior, as the following example illustrates.

Example 3.1. Equation

$$\begin{aligned} x''(t) = {} & \frac{(t^3 - 2\mathrm{e}^t)(t^2 + t\ln x - 2\mathrm{e}^{\frac{t}{2}}(\ln t + \ln x))}{2t(t^2 - \mathrm{e}^t)(\ln t - t)}\, x\Big(\frac{t}{2}\Big) x'(t) \\ & + \frac{t-2}{t(\mathrm{e}^t - t^2)} \bigg(\frac{t(\ln x + t)(x - \mathrm{e}^{-t})}{2(t - \ln t)(\frac{1}{t} - \mathrm{e}^{-t})} + \frac{\mathrm{e}^{\frac{t}{2}}(\ln x + \ln t)(x - \frac{1}{t})}{(\ln t - t)(\mathrm{e}^{-t} - \frac{1}{t})} \bigg) x\Big(\frac{t}{2}\Big) \end{aligned}$$

has solutions $x(t) = \frac{1}{t}$ and $x(t) = \mathrm{e}^{-t}$.

It should be interesting to study the relation between the decay of Kneser solutions and the asymptotic growth of the deviating arguments γ and τ.

Open problem 3. Sufficient conditions ensuring that all bounded solutions of equation

$$(a(t)\Phi(x'))' = b(t) f(x(g(t))), \quad g(t) < t, \tag{3.5}$$

are oscillatory have been given in [5, Corollary 3]. This result is a consequence of some results concerning necessary conditions for the existence of bounded nonoscillatory solutions of (3.5).

It would be interesting to obtain necessary conditions for the existence of Kneser solutions of (1.1) and, as a consequence of such result, to obtain criteria that every bounded solution of (1.1) is oscillatory.

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