On Periodic Boundary Value Problem for a Certain Planar System of Nonlinear Ordinary Differential Equations

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On an interval $[0,\omega]$ we consider the system

$$u_1' = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2 + q_1(t), \quad u_2' = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 + q_2(t) \tag{1}$$

subject to the boundary conditions

$$u_1(0) = u_1(\omega) + c_1, \quad u_2(0) = u_2(\omega) + c_2. \tag{2}$$

Here we suppose that $p_i, q_i \in L([0,\omega])$, $c_i \in \mathbb{R}$, $i = 1, 2$ and

$$\lambda_1 > 0, \quad \lambda_1\lambda_2 = 1. \tag{3}$$

In the linear case, i.e., where $\lambda_1 = 1$ (and $\lambda_2 = 1$), problem (1), (2) as well as its particular case, scalar problem, are studied in sufficient detail. As for the general case, as far as we know, there is still a broad field for further investigations. The aim of the present paper is to fill the existing gap in a certain sense.

Along with (1), (2), we consider also the corresponding "homogeneous" problem

$$u_1' = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \quad u_2' = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1, \tag{1_0}$$

$$u_1(0) = u_1(\omega), \quad u_2(0) = u_2(\omega). \tag{2_0}$$

It has been proved recently in [1] that if (3) holds and (1_0), (2_0) has no non-trivial solution, then for any $q_1, q_2 \in L([0,\omega])$ and $c_1, c_2 \in \mathbb{R}$, problem (1), (2) possesses at least one solution. In other words, the Fredholm property, which is well-known for the linear case, remains true (except uniqueness).

Introduce the definition.

Definition. Let (3) hold and $p_1, p_2 \in L([0,\omega])$. We say that the vector function (p_1, p_2) belongs to the set $V^-(\omega, \lambda_1)$ if for any $(u_1, u_2) \in AC([0,\omega];\mathbb{R}^2)$ such that

$$u_1'(t) = p_1(t)|u_2(t)|^{\lambda_1} \operatorname{sgn} u_2(t), \quad u_2'(t) \ge p_2(t)|u_1(t)|^{\lambda_2} \operatorname{sgn} u_1(t),$$

for a.e. $t \in [0,\omega]$, and

$$u_1(0) = u_1(\omega), \quad u_2(0) \geq u_2(\omega),$$

the inequality

$$u_1(t) \leq 0 \ \text{ for } t \in [0,\omega]$$

is fulfilled.

Remark 1. It is not difficult to verify that if $p_1 \not\equiv 0$ on $[0,\omega]$ and $(p_1,p_2) \in V^-(\omega,\lambda_1)$, then problem (1_0), (2_0) has no non-trivial solutions. Consequently, (1), (2) is solvable, however in spite of linear problem it is not known whether or not the solution of (1), (2) is unique.

Below we suppose also that

$$p_1(t) \geq 0 \ \text{ for a.e. } t \in [0,\omega] \text{ and } p_1 \not\equiv 0 \text{ on } [0,\omega]. \tag{4}$$

The next theorem states that in some cases problem (1), (2) has no more than one solution.

Theorem 1. *Let* (3) *and* (4) *hold,* $(p_1,p_2) \in V^-(\omega,\lambda_1)$, $c \geq 0$, $q \in L([0,\omega])$ *and* $q(t) \geq 0$ *for a.e.* $t \in [0,\omega]$. *Let, moreover,*

$$c + \operatorname{mes}\big\{t \in [0,\omega]: \ q(t) > 0\big\} > 0.$$

Then the problem

$$u_1' = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \quad u_2' = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 - q(t),$$
$$u_1(0) = u_1(\omega), \quad u_2(0) = u_2(\omega) - c$$

is uniquely solvable and its solution (u_1,u_2) *satisfies*

$$u_1(t) > 0 \ \text{ for } t \in [0,\omega].$$

Next, let us present necessary and sufficient conditions for the inclusion $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Theorem 2. *Let* (3) *and* (4) *be fulfilled. Then the inclusion* $(p_1,p_2) \in V^-(\omega,\lambda_1)$ *holds if and only if there exists* $(\gamma_1,\gamma_2) \in AC([0,\omega];\mathbb{R}^2)$ *satisfying*

$$\gamma_1(t) > 0 \ \text{ for } t \in [0,\omega],$$
$$\gamma_1'(t) = p_1(t)|\gamma_2(t)|^{\lambda_1} \operatorname{sgn} \gamma_2(t), \quad \gamma_2'(t) \leq p_2(t)\gamma_1^{\lambda_2}(t) \ \text{ for a.e. } t \in [0,\omega],$$
$$\gamma_1(0) \geq \gamma_1(\omega), \quad \frac{\gamma_2(\omega)}{\gamma_1^{\lambda_2}(\omega)} \geq \frac{\gamma_2(0)}{\gamma_1^{\lambda_2}(0)},$$

and

$$\gamma_1(0) - \gamma_1(\omega) + \frac{\gamma_2(\omega)}{\gamma_1^{\lambda_2}(\omega)} - \frac{\gamma_2(0)}{\gamma_1^{\lambda_2}(0)} + \operatorname{mes}\big\{t \in [0,\omega]: \ \gamma_2'(t) < p_2(t)\gamma_1^{\lambda_2}(t)\big\} > 0.$$

The following corollary follows from Theorem 2 with $(\gamma_1,\gamma_2) \overset{\text{def}}{=} (1,0)$.

Corollary 1. *Let* (3) *and* (4) *hold,* $p_2(t) \geq 0$ *for* $t \in [0,\omega]$, *and* $p_2 \not\equiv 0$ *on* $[0,\omega]$. *Then* $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Corollary 2. *Let* (3) *and* (4) *hold and let there exist* $\varphi \in AC([0,\omega])$ *such that*

$$\int_0^\omega p_1(s)|\varphi(s)|^{\lambda_1} \operatorname{sgn} \varphi(s)\, ds \leq 0, \tag{5}$$

$$\varphi(0) \leq \varphi(\omega), \tag{6}$$

and

$$\Phi(t) \stackrel{\text{def}}{=} \varphi'(t) + \lambda_2 p_1(t)|\varphi_1(t)|^{\lambda_1+1} - p_2(t) \leq 0 \ \ \text{for a.e. } t \in [0,\omega].$$

Let, moreover, either one of inequalities (5) *or* (6) *hold in a strong sense or* $\operatorname{mes}\{t \in [0,\omega] : \Phi(t) < 0\} > 0$. *Then* $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Theorem 1 with suitable choice of vector function (γ_1,γ_2) implies the following efficient conditions for inclusion $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Theorem 3. *Let* (3) *and* (4) *hold,* $p_2 \not\equiv 0$ *on* $[0,\omega]$,

$$\|p_1\|_L \|[p_2]_-\|_L^{\lambda_1} < 2^{\lambda_1+1}, \tag{7}$$

and

$$\|[p_2]_+\|_L > \|[p_2]_-\|_L \Big(1 - \frac{1}{2^{\lambda_1+1}}\, \|p_1\|_L\, \|[p_2]_-\|_L^{\lambda_1}\Big)^{-\lambda_2}. \tag{8}$$

Then the inclusion $(p_1,p_2) \in V^-(\omega,\lambda_1)$ *holds.*

Remark 2. Assumption (8) in Theorem 3 is optimal and cannot be weakened to the assumption

$$\|[p_2]_+\|_L \geq \|[p_2]_-\|_L \Big(1 - \frac{1}{2^{\lambda_1+1}}\, \|p_1\|_L\, \|[p_2]_-\|_L^{\lambda_1}\Big)^{-\lambda_2}. \tag{9}$$

Nevertheless, it is possible to prove the following theorem.

Theorem 4. *Let* (3), (4), (7), *and* (9) *hold. Let, moreover, either*

$$p_1(t) > 0 \ \ \text{for a.e. } t \in [0,\omega]$$

or

$$\lambda_1 < 1 \ \ \text{and} \ \ p_1^{\frac{2}{\lambda_1+1}} \notin L([0,\omega]).$$

Then $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Theorem 5. *Let* (3) *and* (4) *hold,*

$$c \stackrel{\text{def}}{=} \frac{1}{\|p_1\|_L} \int_0^\omega p_2(s)\, ds > 0, \tag{10}$$

and

$$\int_0^\omega [p_2(s) - cp_1(s)]_+\, ds \leq \Big(\frac{c}{\lambda_2}\Big)^{\frac{1}{\lambda_1+1}}. \tag{11}$$

Then $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Example. Let $\omega = 2\pi$, $p_1 \equiv 1$ and $p_2(t) \stackrel{\text{def}}{=} a - b\cos t$ for $t \in [0, \omega]$, where $a > 0$. Then it is clear that

$$\int\limits_0^\omega p_2(s)\,ds = a\omega \ \text{ and } \ c = a$$

with c defined by (10). Assumption (11) has the form $|b| \leq \frac{1}{2}\,(a\lambda_2^{-1})^{\frac{1}{\lambda_1+1}}$. On the other hand, if $a \geq |b|$, then the conditions of Corollary 1 are obviously satisfied. Finally, if (3) holds and

$$|b| \leq \max\Big\{a, \frac{1}{2}\,(a\lambda_2^{-1})^{\frac{1}{\lambda_1+1}}\Big\},$$

then the vector function (p_1, p_2) defined above belongs to the set $V^-(\omega, \lambda_1)$.

References

[1] R. Hakl and M. Zamora, Fredholm-type theorem for boundary value problems for systems of nonlinear functional differential equations. *Bound. Value Probl.* **2014**, 2014:113, 9 pp.