# Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems with the Trivial Averaging of Coefficient Matrix

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Until the middle of the 20th century, the study of periodic solutions of periodic differential systems was based on the hypothesis of the commensurability of the periods of a solution and a system. At the same time, N. D. Papaleksi carried out work on the study of parametric effects on dual-circuit electrical systems. He demonstrated the possibility of excitation of oscillations at a frequency incommensurable with the frequency of changes in the system parameters [8]. In 1950, H. Massera showed that periodic differential systems can have periodic solutions such that the period of a solution is incommensurable with the period of the system. His work [7] laid the foundation for a new direction in the qualitative theory of differential equations which was further developed in the studies of J. Kurzweil and O. Vejvoda [5], N. P. Erugin [2], I. V. Gaĭshun [3], E. I. Grudo [4] and others. Subsequently, such periodic solutions were called strongly irregular [1, p. 16], and the oscillations described by them were called asynchronous. The problem of constructing of asynchronous modes can be formulated as the problem of controlling of the spectrum of irregular oscillations.

First we present the necessary definitions from the theory of almost periodic (on Bohr) functions [6]. Let $f$ be a real continuous function. The function $f$ is called almost periodic if, for an arbitrary positive $\varepsilon$, the set of its $\varepsilon$-almost-periods is relatively dense. Each almost periodic function $f$ has an average value

$$\widehat{f}=\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f(s)\,ds.$$

Put $\widetilde{f}(t)=f(t)-\widehat{f}$. The function $\widetilde{f}$ will be called the oscillating part of an almost periodic function $f$. Note that in contrast to periodic functions, there exist almost periodic functions $\widetilde{f}$ whose integral is not a almost periodic. A real number $\lambda$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \exp(-i\lambda s)f(s)\,ds\neq 0$$

is called the Fourier exponent (or frequency) of an almost periodic function $f$. The set of all frequencies forms the set of Fourier exponents (frequency spectrum) of the function $f$. The module (frequency module) $\mathrm{Mod}(f)$ of an almost periodic function $f$ is the smallest additive group of real numbers containing all the Fourier exponents of this function.

Let $g(t,x)$ be a vector function that is almost periodic in $t$ uniformly with respect to $x$ from some compact set. An almost periodic solution $x(t)$ of the system of ordinary differential equations

$$\dot{x}=g(t,x)$$

will be called strongly irregular if the intersection of the frequency modules of the solution and the right-hand side of the system is trivial, i.e.

$$\mathrm{Mod}(x) \cap \mathrm{Mod}(g) = \{0\}.$$

Let $P(t)$ be a continuous matrix. Denote by $\operatorname{rank}_{col} P$ the column rank of the matrix $P(t)$, i.e. $\operatorname{rank}_{col} P$ is the largest number of its linearly independent columns.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \ \ n \geq 2, \tag{1}$$

where $A(t)$ is a continuous almost periodic $n \times n$-matrix, $B$ is a constant $n \times n$-matrix. We assume that the linear state feedback control

$$u = U(t)x \tag{2}$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ is used, $\mathrm{Mod}(U) \subseteq \mathrm{Mod}(A)$.

The problem of finding a matrix $U(t)$ (the feedback factor) such that the closed-loop system

$$\dot{x} = (A(t) + BU(t))x$$

has a strongly irregular almost periodic solutions with a given frequency spectrum $L$ (the objective set) is called the control problem for the spectrum of irregular oscillations with objective set $L$ (control problem of asynchronous spectrum).

Note first that in the case of a non-singular matrix $B$, the solution of this problem is not difficult. Therefore, we will assume that the matrix $B$ is a singular,

$$\operatorname{rank} B = r < n \ \ (n - r = d).$$

By $B_{d,n}$ and $B_{r,n}$ we denote the matrices consisting of the first $d$ rows and the remaining $r$ rows of the matrix $B$, respectively. One can assume that the first $d$ rows of the matrix $B$ are zero, i.e.,

$$\operatorname{rank} B_{d,n} = 0, \tag{3}$$

because otherwise such a form can be achieved by a linear nonsingular stationary transformation. Note that the rank of the matrix $B_{r,n}$ is equal to $r$ as well.

We will also assume that the matrix $A(t)$ has a zero mean value, i.e.,

$$\widehat{A} = \lim_{T \to \infty} \frac{1}{T} \int\limits_0^T A(s)\, ds = 0, \tag{4}$$

We give conditions for the solvability of the control problem of asynchronous spectrum for system (1).

Let

$$L = \{\lambda_1, \lambda_2, \dots\}$$

be the objective frequency set.

Taking into account the structure of the matrix $B$, we represent the coefficient matrix $A(t)$ in a block form. Let $A_{d,d}(t)$ and $A_{r,d}(t)$ be its left upper and lower blocks, and let $A_{d,r}(t)$ and $A_{r,r}(t)$ be the right upper and lower blocks (the subscripts show the block dimension).

The following theorem holds.

**Theorem.** *Let the first $d$ rows of the matrix $B$ in system* (1) *be zero and the remaining rows be linearly independent, let the coefficient matrix $A(t)$ have a zero mean value, and let the following estimates hold:*

(i) $\operatorname{rank}_{col} A_{d,r} = r_1 < r$;

(ii) $|L| \le [(r - r_1)/2]$.

*Then the control problem for the spectrum of irregular oscillations with objective set L for system* (1) *with feedback* (2) *is solvable.*

**Remark.** *Estimates* (i) *and* (ii) *in the theorem are necessary and sufficient conditions for the solvability of the investigated problem for the class of systems* (1) *under assumptions* (3), (4).

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