Asymptotic Properties of $P_\omega(Y_0, Y_1, 0)$-solutions of Second Order Differential Equations with Rapidly and Regularly Varying Nonlinearities

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

Here, $\alpha_0 \in \{-1; 1\}$, functions $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either an interval $[y_i^0, Y_i]$ or an interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$ we will take $y_i^0 > 0$ or $y_i^0 < 0$, respectively.

We also suppose that the function φ_1 is a regularly varying function of index σ_1 as $y \to Y_1$ $(y \in \Delta_{Y_1})$ [4, pp. 10–15], the function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

From the results obtained in the monograph by V. Maric (see, [3, pp. 91–92, p. 117]) it follows the next lemmas.

Lemma 1. *If the function $\varphi : \Delta_Y \to]0, +\infty[$ is differentiable on Δ_Y and the following condition takes place*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{y\varphi'(y)}{\varphi(y)} = l,$$

then $\varphi(y)$ is normalized slowly or regularly varying function as $y \to Y$ in cases $l = 0$, $l \in R \setminus \{0\}$, respectively, and a rapidly varying function as $y \to Y$ in case $l = \pm\infty$.

Lemma 2. *If the function $\varphi : \Delta_Y \to]0, +\infty[$ is measurable, twice continuously differentiable on Δ_Y and satisfies conditions*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \varphi(y) = Z \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty, \quad \lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{\varphi''(y)\varphi(y)}{(\varphi'(y))^2} = 1,$$

then:

1) *the function φ and its first derivative are rapidly varying functions as $y \to Y$;*

2) *there exists a slowly varying function $l_1 : \Delta_Z \to]0, +\infty[$ as the argument tends to Z (Δ_Z is a one-sided neighborhood of Z) such that*

$$\varphi'(y) = \varphi(y) \cdot l_1(\varphi(y));$$

3) *the function* $F(y) = (\varphi(y))^s$ $(s \in R \setminus \{0\})$ *satisfies the condition*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{F''(y)F(y)}{(F'(y))^2} = 1; \tag{3}$$

4) *the function* $F(y) = \int\limits_{y_0^0}^{y} \varphi(\tau)\,d\tau$, *where*

$$y_0^0 = \begin{cases} y_0 & as \displaystyle\int\limits_{y_0}^{Y} \varphi(\tau)\,d\tau = +\infty, \\ Y & as \displaystyle\int\limits_{y_0}^{Y} \varphi(\tau)\,d\tau < +\infty, \end{cases}$$

satisfies condition (3).

Lemma 3. *If the function* $\varphi : \Delta_Y \to]0,+\infty[$ *satisfies conditions* (2), *the function* $L : \Delta_Y \to]0,+\infty[$ *is a slowly varying function as* $y \to Y$ $(y \in \Delta_Y)$, *then*

$$\int\limits_{y_0^0}^{y} L(\tau)\varphi(\tau)\,d\tau \sim L(y) \int\limits_{y_0^0}^{y} \varphi(\tau)\,d\tau \quad as \;\; y \to Y,$$

where

$$y_0^0 = \begin{cases} y_0 & as \displaystyle\int\limits_{y_0}^{Y} L(\tau)\varphi(\tau)\,d\tau = +\infty, \\ Y & as \displaystyle\int\limits_{y_0}^{Y} L(\tau)\varphi(\tau)\,d\tau < +\infty, \end{cases} \qquad y_0 \in \Delta_Y.$$

Lemma 4. *If* $\varphi_0 : \Delta_{Y_0} \to]0,+\infty[$ *is a rapidly varying function as the argument tends to* Y_0, *the function* $\varphi_1 : \Delta_{Y_1} \to \Delta_{Y_0}$ *satisfies the condition* $\lim\limits_{\substack{y \to Y_1 \\ y \in \Delta_Y}} \varphi_1(y) = Y_0$ *and is a regularly varying function of index* $\sigma \neq 0$ *as the argument tends to* Y_1, *then the function* $\varphi_0(\varphi_1)$ *is also a rapidly varying function as the argument tends to* Y_1.

Lemma 5. *If the rapidly varying as* $y \to Y$ *function* $\varphi : \Delta_Y \to]0,+\infty[$ *is strictly monotone on* Δ_Y *and satisfies the conditions*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \varphi(y) = Z \in \{0,+\infty\}, \quad \varphi(\Delta_Y) = \Delta_Z,$$

where Δ_Z *is one-sided neighborhood of* Z, *then the function* $\varphi^{-1} : \Delta_Z \to \Delta_Y$ *is a slowly varying function as the argument tends to* Z.

Definition 1. The solution y of equation (1), defined on the interval $[t_0,\omega[\subset [a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the following conditions take place

$$y^{(i)} : [t_0,\omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \;\; (i = 0,1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In this work we establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) in case $\lambda_0 = 0$ and find asymptotic representations of such solutions and its first order derivatives as $t \uparrow \omega$.

The main result of the work is obtained under the assumption that for $P_\omega(Y_0, Y_1, 0)$-solutions of equation (1) there exist the next finite or infinite limit

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) y''(t)}{y'(t)}\,.$$

According to the properties of such solutions (see, for example, [1]) we have

$$\begin{aligned} &\lim_{t\uparrow\omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = 0,\\ &\lim_{t\uparrow\omega} \frac{\pi_\omega(t) y''(t)}{y'(t)} = -1. \end{aligned} \tag{4}$$

From (4) it follows that function $y'(t)$ is a normalized regularly varying function of index (-1) as $t \uparrow \omega$, that means it can be represented in the form

$$y'(t) = |\pi_\omega(t)|^{-1} L_1(t),$$

where $L_1(t) : [t_0, \omega[\to] - \infty, +\infty[$ is a normalized slowly varying function as $t \uparrow \omega$ [4, pp. 10–15]. It follows that

$$\lim_{t\uparrow\omega} \frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(t)|} = Y_1.$$

From the fact that the function L_1 is a normalized slowly varying function, it follows that the function $L_1(t(z))$, where $t(z)$ is the inverted function to the function $z(t) = \frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(t)|}$, is also a normalized slowly varying function as $t \uparrow \omega$ because it is a composition of slow and regularly varying functions.

Let us introduce in the following notations.

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty,\\ t-\omega & \text{if } \omega < +\infty, \end{cases} \quad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi(y) = \int_{A_\omega}^{y} |\varphi_0(z)|^{\frac{1}{\sigma_1 - 1}}\, dz, \quad A_\omega = \begin{cases} y_0^0 & \text{if } \displaystyle\int_{y_0^0}^{Y_0} |\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\, dz = \pm\infty,\\ Y_0 & \text{if } \displaystyle\int_{y_0^0}^{Y_0} |\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\, dz = const, \end{cases}$$

$$\mu_0 = \operatorname{sign}(\varphi_0'(y)), \quad Z_0 = \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \Phi(y).$$

From the indicated conditions onto the function φ_0 we have

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \Phi(y) \in \{0, +\infty\}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\Phi''(y)\cdot\Phi(y)}{(\Phi'(y))^2} = 1.$$

It follows from this that, like the function φ_0, the function Φ is also a rapidly varying function when the argument tends to Y_0 [4, pp. 10–15]. In addition, the following lemma takes place.

Lemma 6.

1)
$$\Phi(y)=(\sigma_1-1)\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)] \ \ as \ \ y\to Y_0 \ \ (y\in\Delta_{Y_0}),$$

from which we have
$$\mu_0\cdot\operatorname{sign}(\Phi(y))=\operatorname{sign}(\sigma_1-1) \ \ as \ \ y\in\Delta_{Y_0}.$$

2) *The function* $\Phi^{-1}(z)\cdot\frac{\Phi'(\Phi^{-1}(z))}{z}$ *is a slowly varying function as* $z\to Z_0$.

Proof. Statement 1) of the lemma follows from the conditions on the function φ_0.

Let us prove statement 2). We have
$$\lim_{z\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1.$$

So,
$$\lim_{z\to Z_1}\frac{z\cdot(\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z})'}{\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}}=\lim_{y\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}-1=0.$$

The last one means that the function $\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is a slowly varying function as $z\to Z_1$. And the function $\Phi_1^{-1}(z)$ is a slowly varying as $z\to Z_1$ like an inverse function to the rapidly varying one. □

Let's introduce the additional notations.
$$I(t)=\operatorname{sign}(y_1^0)\cdot\int\limits_{B_\omega^2}^{t}\bigg|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\bigg|^{\frac{1}{1-\sigma_1}}\,d\tau,$$
$$B_\omega^2=\begin{cases}\omega & \text{if} \ \ \displaystyle\int\limits_b^\omega\bigg|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\bigg|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\\ b & \text{if} \ \ \displaystyle\int\limits_b^\omega\bigg|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\bigg|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\end{cases} \qquad b\in[a;\omega[\,.$$

Definition 2. We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S as $z\to Y$ if for any continuous differentiable normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$ the next relation is valid
$$\theta(zL(z))=\theta(z)(1+o(1)) \ \ \text{as} \ \ z\to Y \ \ (z\in\Delta_Y).$$

Conditions S are satisfied, for example, for such functions as $\ln|y|$, $|\ln|y||^\mu$ $(\mu\in R)$, $\ln\ln|y|$.

The following theorem is valid.

Theorem 1. *Let $\sigma_1\neq 1$, the function θ_1 satisfy the condition S. For the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1), for which the following finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$ *exists, the* *following conditions are necessary*

$$\alpha_0\pi_\omega(t)y_1^0 < 0 \;\; as \;\; t \in [a;\omega[\,,$$

$$\lim_{t\uparrow\omega} \frac{y_1^0}{|\pi_\omega(t)|} = Y_1, \tag{5}$$

$$\lim_{t\uparrow\omega} I_2(t) = Z_0, \quad \mu_0(\sigma_1 - 1)I_2(t) > 0 \;\; as \;\; t \in]b;\omega[\,, \tag{6}$$

$$\lim_{t\uparrow\omega} \frac{I_2'(t)\pi_\omega(t)}{\Phi_2'(\Phi_2^{-1}(I_2(t)))\Phi_2^{-1}(I_2(t))} = 0. \tag{7}$$

For each such solution the next asymptotic representations take place as $t \uparrow \omega$:

$$\Phi_0(y(t)) = I_2(t)[1+o(1)], \quad \frac{y'(t)\Phi_0'(y(t))}{\Phi_2(y(t))} = \frac{I_2'(t)}{I_2(t)}[1+o(1)]. \tag{8}$$

Theorem 2. *Let $\sigma_1 \neq 1$, the function θ_1 satisfy the condition S, the function $\frac{\pi_\omega(t)\cdot I'(t)}{I(t)}$ be a normalized slowly varying function as $t \uparrow \omega$, the function $(\frac{\Phi'(y)}{\Phi(y)})$ be a regularly varying function of some real index as $y \to Y_0$ $(y \in \Delta_{Y_0})$. Then in case either*

$$0 < \Big|\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I_2'(t)}{I_2(t)}\Big| < +\infty, \tag{9}$$

or

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \pm\infty, \quad \mu_0\alpha_0 < 0, \tag{10}$$

conditions (5)–(7) *are sufficient for the existence of $P_\omega(Y_0, Y_1, 0)$-solutions of equation* (1)*, for which the finite or infinite limit $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)}$ exists.*

During the proof of Theorem 2, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system has real eigenvalues of different signs. We obtain that for this system of differential equations all the conditions of Theorem 2.2 in [2] take place. According to this theorem, the system has a one-parameter family of solutions $\{z_i\}_{i=1}^2 : [x_1, +\infty[\, \to \mathbb{R}^2$ $(x_1 \geq x_0)$, that tends to zero as $x \to +\infty$.

Any solution of the family gives raise to such a solution y of equation (1) that, together with its first derivative, admits the asymptotic images (8) as $t \uparrow \omega$. From these images and conditions (5)–(7), (9), (10) it follows that these solutions are $P_\omega(Y_0, Y_1, 0)$-solutions.

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