On the Solvability of Focal Boundary Value Problems for Higher-Order Linear Functional Differential Equations

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We obtain sharp solvability conditions for focal boundary value problems for higher-order linear functional differential equations with functional operators under integral and point-wise restrictions.

Consider the focal boundary value problem

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t)+(Tx)(t)=f(t), & t\in[0,1],\\ x^{(i)}(0)=0, & i=0,\dots,k-1,\\ x^{(j)}(1)=0, & j=k,\dots,n-1, \end{cases} \tag{0.1}$$

where $k\in\{1,2,\dots,n-1\}$, $n\geq 2$, $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is a linear boundary operator, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the spaces of continuous and integrable real functions on the interval $[0,1]$ (wish usual norms).

The problems of solving various focal boundary value problems for linear and nonlinear ordinary differential equations and functional differential equations arise in many studies of physical, chemical, and biological processes [1, 2, 8, 13, 15].

For the zero operator T, the boundary value problem

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t)=f(t), & t\in[0,1],\\ x^{(i)}(0)=0, & i=0,\dots,k-1,\\ x^{(j)}(1)=0, & j=k,\dots,n-1 \end{cases}$$

has a unique solution $x(t)=\int\limits_0^1 G(t,s)f(s)\,ds$, $t\in[0,1]$, where the Green function (see, for example, [8])

$$G(t,s)=\frac{1}{(n-k-1)!}\,\frac{1}{(k-1)!}\int\limits_0^{\min(t,s)}(s-\tau)^{n-k-1}(t-\tau)^{k-1}\,d\tau,\ \ t,s\in[0,1],$$

is non-negative.

1 Integral restrictions

The following simple assertion is a corollary of the Banach fixed-point theorem and the Fredholm property of the boundary value problem.

Assertion 1.1. *If* $\|T\|_{\mathbf{C}\to\mathbf{L}}\leq(n-1)(n-k-1)!(k-1)!$, *then problem* (0.1) *is uniquely solvable.*

Definition 1.1. A linear operator $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

In this work, we weaken the solvability conditions from Assertion 1.1 in the case of positive operator T. For some other boundary value problems similar unimprovable conditions are obtained by R. Hakl, A. Lomtatidze, S. Mukhigulashvili, B. Půža, J. Šremr, and others [6, 9–12].

The norm of a positive operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is defined by the equality

$$\|T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(t)\,dt,$$

where $\mathbf{1}$ is the unit function.

Theorem 1.1. *Let a non-negative number $\mathcal{T}$ be given. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if the following inequality is valid:*

$$\mathcal{T} \le \min_{0<t<1,\,0<s<1} \frac{G(t,1)+G(1,s)+2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1)-G(t,1)G(1,s)}\,.$$

Remark 1.1. In (1.1), the expression $G(t,s)G(1,1)-G(t,1)G(1,s)$ is positive for all $t,s \in (0,1)$ because of the kernel $G(t,s)$ is totally positive (see, for example, [7, 14]).

The proof of Theorem 1.1 is based on the following lemma.

Lemma 1.1 ([3])**.** *Let a non-negative number $\mathcal{T}$ be given. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if for all numbers c, d, τ_1, τ_2, $\mathcal{T}_1$, $\mathcal{T}_2$ satisfied the conditions*

$$\begin{gathered} c,d \in [0,1], \ \ 0 \le \tau_1 \le \tau_2 \le 1, \\ \mathcal{T}_1 \ge 0, \ \ \mathcal{T}_2 \ge 0, \ \ \mathcal{T}_1+\mathcal{T}_2 \le \mathcal{T}, \end{gathered}$$

the inequality

$$1+\mathcal{T}_1G(\tau_1,c)+\mathcal{T}_2G(\tau_2,d)+\mathcal{T}_1\mathcal{T}_2\big(G(\tau_1,c)G(\tau_2,d)-G(\tau_2,c)G(\tau_1,d)\big) \ge 0$$

is fulfilled.

Theorem 1.2. *Let a non-negative number $\mathcal{T}$ be given and $n = 2k$. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if the following inequality is valid:*

$$\mathcal{T} \le \frac{2((n/2-1)!)^2}{\max\limits_{0<t<1} \big(\frac{t^{(n-1)/2}}{n-1} - \int_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau\big)} \equiv \mathcal{T}_n.$$

For $n=2$, $n=4$, $n=6$, the numbers $\mathcal{T}_n$ can be calculated exactly. We have

$$\begin{aligned} \mathcal{T}_2 &= 8, \\ \mathcal{T}_4 &= 66+30\sqrt{5} \approx 133.1, \\ \mathcal{T}_6 &= \frac{8}{\frac{t^{5/2}}{5} - \frac{t_6^3(t_6^2-5t_6+10)}{30}} \approx 2610.5, \end{aligned}$$

where

$$t_6 = \Big(\frac{C_1-1-\sqrt{27+22/C_1-C_1^2}}{4}\Big)^2,$$
$$C_1 = \sqrt{2C_2+9+48/C_2}, \quad C_2 = \sqrt[3]{124+4\sqrt{97}}\,.$$

For even $n \ge 8$, we obtain sufficient solvability conditions.

Corollary 1.1. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq \frac{(n^2-9)(n^2-1)((n/2-1)!)^2}{3+(n-2)(\frac{n-7}{n-3})^{\frac{n+1}{2}}},$$

then the boundary value problem (0.1) *is uniquely solvable.*

Corollary 1.2. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq e^2(n-3)^3((n/2-1)!)^2, \tag{1.1}$$

then the boundary value problem (0.1) *is uniquely solvable.*

Remark 1.2. The sufficient condition in Corollary 1.2 is sharp. The constant e^2 and the exponents cannot be increased in (1.1). Inequality (1.1) significantly improves the solvability condition from Assertion 1.1 (the constant in the solvability conditions is increased approximately $(en)^2$ times for large n).

2 Point-wise restrictions

Consider problem (0.1) for $k = n-1$,

$$\begin{cases} x^{(n)}(t) - (Tx)(t) = f(t), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0,\dots,n-2, \\ x^{(n-1)}(1) = 0. \end{cases} \tag{2.1}$$

Assertion 2.1. *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear bounded operator. If*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} |(T\mathbf{1})(t)| < (n-2)!n,$$

then problem (2.1) *is uniquely solvable.*

We can improve this assertion for positive operators T.

Lemma 2.1 ([3, Lemma 2.19], [4, Lemma 2], [5, Lemma 1])**.** *Let a non-negative function $p \in L[0,1]$ be given. Problem* (2.1) *is uniquely solvable for all positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ satisfied the equality $T\mathbf{1} = p$ if and only if the focal boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0,\dots,n-2, \\ x^{(n-1)}(1) = 0 \end{cases}$$

has only the trivial solution for all points $t_1 \leq t_2$, $t_1, t_2 \in [0,1]$ and for all non-negative functions $p_1, p_2 \in L[0,1]$ such that $p_1 + p_2 = p$.

Define

$$k(t) \equiv 1 + P\Big(1 - \frac{t}{n}\Big)\frac{t^{n-1}}{(n-1)!}, \quad t \in [0,1],$$

where P is a constant,

$$G_1(t,s) \equiv \begin{cases} \dfrac{t^{n-1}-(t-s)^{n-1}}{(n-1)!}, & 1 \geq t \geq s \geq 0, \\ \dfrac{t^{n-1}}{(n-1)!}, & 1 \geq s > t \geq 0. \end{cases}$$

Theorem 2.1. *Let a non-negative number P be given. Then the focal boundary value problem* (2.1) *is uniquely solvable for all positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le P$$

if and only if the inequality

$$k(t_2) + P\int\limits_s^1 \big(G_1(t_2,\tau)k(t_1) - G_1(t_1,\tau)k(t_2)\big)\,d\tau > 0$$

is fulfilled for all $0 \le t_1 \le t_2 \le 1$ and all $s \in (0, t_2]$.

We obtain some sufficient solvability conditions for the simplest functional differential equations with one concentrated argument.

Corollary 2.1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative coefficient, $h : [0,1] \to [0,1]$ be a measurable deviated argument.*

Then for $n = 2$, the focal boundary value problem

$$\begin{cases} x^{(n)}(t) = p(t)x(h(t)) + f(t), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0,\dots,n-2, \\ x^{(n-1)}(1) = 0 \end{cases} \tag{2.2}$$

is uniquely solvable if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 16, \quad p(t) \not\equiv 16,$$

where the constant "16" is unimprovable.

For $n = 3$, problem (2.2) *is uniquely solvable if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 58.$$

For $n = 4$, problem (2.2) *is uniquely solvable if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 270.$$

Remark 2.1. It seems that for $n = 2$ the best constants "8" and "16" in Theorem 1.2 and Corollary 2.1 are known (see, for example, [3, p. 109] for integral restriction). However, as we know, for higher-order functional differential equations these results are new.

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