Generalization of Perron's and Vinograd's Examples of Lyapunov Exponents Instability to Linear Differential Systems with Parametric Perturbations

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For a given positive integer n let us denote by $\mathcal{M}_n$ the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

defined on the time semi-axis $\mathbb{R}_+$ with continuous bounded coefficients. Let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ denote the Lyapunov exponents [6, p. 561], [1, p. 38] of the system (1). Besides, we denote by $\mathcal{R}_n$ the subclass of the class $\mathcal{M}_n$ consisting of Lyapunov regular systems [6, p. 563], [1, p. 61]. In what follows, we identify the system (1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$ or $A \in \mathcal{R}_n$.

In the paper [7] O. Perron constructed an example of a system $A \in \mathcal{M}_2$ with negative Lyapunov exponents for which there exists an exponentially decaying perturbation $Q : \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ such that the largest Lyapunov exponent of the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \;\; x \in \mathbb{R}^2, \;\; t \in \mathbb{R}_+,$$

is positive. Put differently, the Lyapunov exponents, which are responsible for the stability, are not stable themselves (even under those perturbations of a system's coefficient matrix that decay exponentially).

As a result of Perron's example the problem naturally arises of finding wide enough subclasses of the class $\mathcal{M}_n$ consisting of the systems whose Lyapunov exponents are invariant under vanishing at infinity perturbations of the coefficient matrix. It was a long-standing conjecture that the class $\mathcal{R}_n$ of Lyapunov regular systems possesses the desired property. The conjecture was based essentially on the fundamental result by Lyapunov which claims that if a nonlinear system (with natural restrictions on the right-hand side) has a regular first approximation system and the latter is conditionally exponentially stable, then so is the zero solution of the original system (with the same dimension of the stable manifold and asymptotic exponent) [6, pp. 577–579]. Nevertheless, in the paper [8] R. E. Vinograd provided an example of a system $A \in \mathcal{R}_2$ whose Lyapunov exponents change under some vanishing at infinity perturbation of its coefficient matrix (the Lyapunov exponents of a regular system are invariant under exponentially decaying perturbations of its coefficient matrix, which is implied by Bogdanov–Grobman theorem [5, p. 188]).

Let M be a metric space. Let us introduce the classes $\mathcal{E}_n(M)$ and $\mathcal{Z}_n(M)$ of jointly continuous matrix-valued functions $Q(\,\cdot\,,\,\cdot\,)\colon \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$. The class $\mathcal{E}_n(M)$ consists of the functions $Q(\,\cdot\,,\,\cdot\,)$ exponentially decaying as $t \to +\infty$ with a uniform exponent with respect to $\mu \in M$:

$$\varlimsup_{t\to+\infty} t^{-1} \ln \|Q(t,\mu)\| < const < 0,$$

and the class $\mathcal{Z}_n(M)$ consists of the functions $Q(\,\cdot\,,\,\cdot\,)$ vanishing at infinity uniformly in $\mu \in M$:

$$\lim_{t\to+\infty} \sup_{\mu\in M} \|Q(t,\mu)\| = 0.$$

Generalizing the situation considered in examples of Perron and Vinograd, for each system $A \in \mathcal{M}_n$, let us define the class $\mathcal{P}_n(A; M)$ consisting of the families

$$\dot{x} = (A(t) + Q(t,\mu))x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{2}$$

of linear differential systems, where $\mu \in M$ is a parameter and $Q(\,\cdot\,,\,\cdot\,) \in \mathcal{E}_n(M)$. Next, for each $A \in \mathcal{R}_n$ we define the class $\mathcal{V}_n(A; M)$ to consist of those families (2) in which $Q(\,\cdot\,,\,\cdot\,) \in \mathcal{Z}_n(M)$. Therefore, fixing a value of the parameter $\mu \in M$ in the family (2) we obtain a linear differential system with continuous coefficients bounded on the semi-axis. Let $\lambda_1(\mu; A+Q) \leqslant \cdots \leqslant \lambda_n(\mu; A+Q)$ stand for the Lyapunov exponents of this system. Thus for each $k = \overline{1,n}$ we get the function $\lambda_k(\,\cdot\,; A)\colon M \to \mathbb{R}$, which is called the k-th Lyapunov exponent of the family (2), and the vector function $\Lambda(\,\cdot\,; A+Q) : M \to \mathbb{R}^n$ defined by $\Lambda(\mu; A+Q) = (\lambda_1(\mu; A+Q), \dots, \lambda_n(\mu; A+Q))$, which is called the spectrum of the Lyapunov exponents of the family (2).

We state the problems to be solved as follows: for each $n \in \mathbb{N}$ and every metric space M completely describe the classes of vector functions

$$\begin{aligned}\mathcal{P}_n(M) &= \big\{\Lambda(\,\cdot\,; A+Q) \mid \; A \in \mathcal{M}_n, \; Q \in \mathcal{E}_n(M)\big\},\\ \mathcal{V}_n(M) &= \big\{\Lambda(\,\cdot\,; A+Q) \mid \; A \in \mathcal{R}_n, \; Q \in \mathcal{Z}_n(M)\big\}.\end{aligned}$$

Solutions to these problems will contain as special cases examples of Perron and Vinograd, respectively. If $n = 1$, then the descriptions of the above classes immediately follow from the definition of the Lyapunov exponent – for any metric space M both the classes $\mathcal{P}_1(M)$ and $\mathcal{V}_1(M)$ coincide with the class of constant functions $M \to \mathbb{R}$. Therefore, from now on, we assume that $n \geqslant 2$.

Let a vector function $f(\,\cdot\,) = (f_1(\,\cdot\,), \dots, f_n(\,\cdot\,)) : M \to \mathbb{R}^n$ belong to the class $\mathcal{P}_n(M)$ or to the class $\mathcal{V}_n(M)$. Let us state three properties of the vector function $f(\,\cdot\,)$ that it must satisfy (below these properties are numbered as 1), 2), 3)). One of the properties is trivially implied by the very definition of this vector function: 1) for every $\mu \in M$ the inequalities $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ hold. Another property follows from the fact that a matrix-valued function A is bounded on the time semi-axis and for every $\mu \in M$, a perturbation matrix $Q(\,\cdot\,,\mu)$ vanishes at infinity: 2) the vector function $f(\,\cdot\,)$ is bounded on M. For example, $|\Lambda(\mu; A+Q)| \leqslant n \sup\{\|A(t)\| \mid \; t \in \mathbb{R}_+\}$ for all $\mu \in M$. Before stating the third property let us recall that a function $g : M \to \mathbb{R}$ is said [4, p. 267] to be of the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$ the preimage $g^{-1}([r,+\infty))$ of the half-interval $[r,+\infty)$ is a G_δ-set of the metric space M. As follows from the paper [2], in which a complete description is obtained for the spectra of the Lyapunov exponents of general parametric families of linear differential systems continuous in the parameter uniformly in $t \in \mathbb{R}_+$, the property 3) is true: the components $f_k(\,\cdot\,)$ of the vector function $f(\,\cdot\,)$ are of the class $({}^*, G_\delta)$.

Theorem 1. *Let M be a metric space, $n \geqslant 2$ an integer, and a vector function $f\colon M \to \mathbb{R}^n$ satisfy the properties* 1)*–*3)*. Then there exist a system $A \in \mathcal{M}_n$ and a matrix-valued function $Q \in \mathcal{E}_n(M)$ such that the spectrum of the Lyapunov exponents of the family* (2) *coincides with the function f, i.e. $\Lambda(\mu; A+Q) = f(\mu)$ for all $\mu \in M$.*

Theorem 2. *Let M be a metric space, $n \geqslant 2$ an integer, and a vector function $f : M \to \mathbb{R}^n$ satisfy the properties* 1)*–*3)*. Then there exist a Lyapunov regular system $A \in \mathcal{R}_n$ and a matrix-valued function $Q \in \mathcal{Z}_n(M)$ such that the spectrum of the Lyapunov exponents of the family* (2) *coincides with the function f, i.e. $\Lambda(\mu; A+Q) = f(\mu)$ for all $\mu \in M$.*

Thus, from the said above it follows that the classes $\mathcal{P}_n(M)$ and $\mathcal{V}_n(M)$ are identical, and their common complete description is contained in the following

Theorem 3. *For any $n \geqslant 2$ and every metric space M, a vector function $(f_1, \dots, f_n) : M \to \mathbb{R}^n$ belongs to the class $\mathcal{P}_n(M)$ (to the class $\mathcal{V}_n(M)$) if and only if it satisfies the properties 1)–3). For each metric space M the class $\mathcal{P}_1(M)$ (the class $\mathcal{V}_1(M)$ coincides with the class of constant functions $M \to \mathbb{R}$.*

Note that if M is a segment of the real line, then in Theorems 1–3 above one can choose a matrix-valued function $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ to be analytical in $\mu \in M$ for each $t \in \mathbb{R}_+$.

Recall that a subset of a metric space M is said to be an $F_{\sigma\delta}$-set if it can be expressed as countable intersection of F_σ sets in M. The latter, in turn, are those which can be represented as countable unions of closed sets in M [4, p. 96]. Combining Theorem 2 above with [3, Corollary 2] we arrive at the following

Corollary. *Let an integer $n \geqslant 2$ and a metric space M be given. Then for any $F_{\sigma\delta}$-set S in M there exist a Lyapunov regular system $A \in \mathcal{R}_n$ and a matrix-valued function $Q \in \mathcal{Z}_n(M)$ such that S is the set of those $\mu \in M$ for which the system (2) is Lyapunov regular.*

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