On Linear Boundary-Value Problems for Differential Systems in Sobolev spaces

Olena Atlasiuk, Vladimir Mikhailets

Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv, Ukraine
E-mails: hatlasiuk@gmail.com; mikhailets@imath.kiev.ua

Let a finite interval $[a,b]\subset\mathbb{R}$ and parameters $\{m,n,r\}\subset\mathbb{N}$, $1\leqslant p\leqslant\infty$, be given. By $W_p^n=W_p^n([a,b];\mathbb{C}):=\{y\in C^{n-1}[a,b]:\ y^{(n-1)}\in AC[a,b],\ y^{(n)}\in L_p[a,b]\}$ we denote a complex Sobolev space and set $W_p^0:=L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n,p}=\sum_{k=0}^{n-1}\|y^{(k)}\|_p+\|y^{(n)}\|_p,$$

where $\|\cdot\|_p$ is the norm in the space $L_p([a,b];\mathbb{C})$. Similarly, by $(W_p^n)^m:=W_p^n([a,b];\mathbb{C}^m)$ and $(W_p^n)^{m\times m}:=W_p^n([a,b];\mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^n.

We consider the following linear boundary-value problem

$$Ly(t):=y'(t)+A(t)y(t)=f(t),\ \ t\in(a,b), \tag{1}$$

$$By=c, \tag{2}$$

where the matrix-valued function $A(\cdot)\in(W_p^{n-1})^{m\times m}$, the vector-valued function $f(\cdot)\in(W_p^{n-1})^m$, the vector $c\in\mathbb{C}^r$, the linear continuous operator

$$B:(W_p^n)^m\to\mathbb{C}^r \tag{3}$$

are arbitrarily chosen; and the vector-valued function $y(\cdot)\in(W_p^n)^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution of the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot)\in(W_p^n)^m$ satisfying equation (1) almost everywhere on (a,b) (everywhere for $n\geq 2$) and equality (2) specifying r scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^n)^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^{n-1})^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multi-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives of the unknown functions of the order $k\leqslant n$.

It is known that, for $1\leq p<\infty$, every operator B in (3) admits a unique analytic representation

$$By=\sum_{k=0}^{n-1}\alpha_k y^{(k)}(a)+\int_a^b\Phi(t)y^{(n)}(t)\,\mathrm{d}t,\ \ y(\cdot)\in(W_p^n)^m,$$

where the matrices $\alpha_k\in\mathbb{C}^{r\times m}$ and the matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{r\times m})$, $1/p+1/p'=1$.

For $p=\infty$ this formula also defines an operator $B\in L((W_\infty^n)^m;\mathbb{C}^r)$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

We rewrite the inhomogeneous boundary-value problem (1), (2) in the form of a linear operator equation $(L,B)y=(f,c)$, where (L,B) is a linear operator in the pair of Banach spaces

$$(L,B):(W_p^n)^m\to(W_p^{n-1})^m\times\mathbb{C}^r. \tag{4}$$

Recall that a linear continuous operator $T:X\to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind}T:=\dim\ker T-\dim(Y/T(X))$$

is finite.

Theorem 1. *The linear operator* (4) *is a bounded Fredholm operator with index* $m-r$*.*

Theorem 1 allows the next refinement.

By $Y(\,\cdot\,)\in(W_p^n)^{m\times m}$ we denote a unique solution of the linear homogenous matrix equation $(LY)(t)=O_m$, $Y(a)=I_m$, where O_m is the $(m\times m)$ zero matrix, and I_m is the $(m\times m)$ identity matrix.

Definition 1. A rectangular numerical matrix $M(L,B)\in\mathbb{C}^{m\times r}$ is characteristic for the boundary-value problem (1), (2) if its j-th column is the result of the action of the operator B on the j-th column of the matricant $Y(\,\cdot\,)$.

Here m is the number of scalar differential equations of the system (1), and r is the number of scalar boundary conditions.

Theorem 2. *The dimensions of the kernel and cokernel of the operator* (4) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix* $M(L,B)$ *respectively.*

Theorem 2 implies a criterion for the invertibility of the operator (4).

Corollary 1. *The operator* (L,B) *is invertible if and only if* $r=m$ *and the matrix* $M(L,B)$ *is nondegenerate.*

Let us consider parameterized by number $\varepsilon\in[0,\varepsilon_0)$, $\varepsilon_0>0$, linear boundary-value problem

$$L(\varepsilon)y(t;\varepsilon):=y'(t;\varepsilon)+A(t;\varepsilon)y(t;\varepsilon)=f(t;\varepsilon),\ \ t\in(a,b), \tag{5}$$

$$B(\varepsilon)y(\,\cdot\,;\varepsilon)=c(\varepsilon), \tag{6}$$

where for every fixed ε the matrix-valued function $A(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^{m\times m}$, the vector-valued function $f(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^m$, the vector $c(\varepsilon)\in\mathbb{C}^m$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon):(W_p^n)^m\to\mathbb{C}^m$, and the solution (the unknown vector-valued function) $y(\,\cdot\,;\varepsilon)\in(W_p^n)^m$.

It follows from Theorem 2 that the boundary-value problem (5), (6) is a Fredholm one with index zero.

Definition 2. A solution of the boundary-value problem (5), (6) continuously depends on the parameter ε for $\varepsilon=0$ if the following conditions are satisfied:

- $(*)$ there exists a positive number $\varepsilon_1<\varepsilon_0$ such that, for any $\varepsilon\in[0,\varepsilon_1)$ and arbitrary right-hand sides $f(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^m$ and $c(\varepsilon)\in\mathbb{C}^m$ this problem has a unique solution $y(\,\cdot\,;\varepsilon)$ that belongs to the space $(W_p^n)^m$;
- $(**)$ the convergence of the right-hand sides $f(\,\cdot\,;\varepsilon)\to f(\,\cdot\,;0)$ in $(W_p^{n-1})^m$ and $c(\varepsilon)\to c(0)$ in $\mathbb{C}^m$ as $\varepsilon\to0+$ implies the convergence of the solutions $y(\,\cdot\,;\varepsilon)\to y(\,\cdot\,;0)$ in $(W_p^n)^m$.

Consider the following conditions as $\varepsilon \to 0+$:

(0) limiting homogeneous boundary-value problem

$$L(0)y(t,0)=0,\ \ t\in(a,b),\quad B(0)y(\,\cdot\,,0)=0$$

has only the trivial solution;

(I) $A(\,\cdot\,,\varepsilon)\to A(\,\cdot\,,0)$ in the space $(W_p^{n-1})^{m\times m}$;

(II) $B(\varepsilon)y\to B(0)y$ in $\mathbb{C}^m$ for any $y\in(W_p^n)^m$.

Theorem 3. *A solution of the boundary-value problem* (5), (6) *continuously depends on the parameter ε for $\varepsilon=0$ if and only if it satisfies condition (0) and the conditions* (I) *and* (II).

Consider the following quantities:

$$\big\|y(\,\cdot\,;0)-y(\,\cdot\,;\varepsilon)\big\|_{n,p}, \tag{7}$$

$$\widetilde{d}_{n-1,p}(\varepsilon):=\big\|L(\varepsilon)y(\,\cdot\,;0)-f(\,\cdot\,;\varepsilon)\big\|_{n-1,p}+\big\|B(\varepsilon)y(\,\cdot\,;0)-c(\varepsilon)\big\|_{\mathbb{C}^m}, \tag{8}$$

where (7) is the error and (8) is the discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ of the boundary-value problem (5), (6) if $y(\,\cdot\,;\varepsilon)$ is its exact solution and $y(\,\cdot\,;0)$ is an approximate solution of the problem.

Theorem 4. *Suppose that the boundary-value problem* (5), (6) *satisfies conditions (0), (I) and (II). Then there exist the positive quantities $\varepsilon_2<\varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon\in(0,\varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\,\widetilde{d}_{n-1,p}(\varepsilon)\le\big\|y(\,\cdot\,;0)-y(\,\cdot\,;\varepsilon)\big\|_{n,p}\le\gamma_2\,\widetilde{d}_{n-1,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend of $y(\,\cdot\,;\varepsilon)$ and $y(\,\cdot\,;0)$.

According to this theorem, the error and discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ of the boundary-value problem (5), (6) have the same order of smallness.

For any $\varepsilon\in[0,\varepsilon_0)$, $\varepsilon_0>0$, we associate with the system (5) multi-point Fredholm boundary condition

$$B(\varepsilon)y(\,\cdot\,,\varepsilon)=\sum_{j=0}^{r}\sum_{k=1}^{\omega_j(\varepsilon)}\sum_{l=0}^{n}\beta_{j,k}^{(l)}(\varepsilon)y^{(l)}(t_{j,k}(\varepsilon),\varepsilon)=q(\varepsilon), \tag{9}$$

where the numbers $\{r,\omega_j(\varepsilon)\}\subset\mathbb{N}$, vectors $q(\varepsilon)\in\mathbb{C}^m$, matrices $\beta_{j,k}^{(l)}(\varepsilon)\in\mathbb{C}^{m\times m}$, and points $\{t_j,t_{j,k}(\varepsilon)\}\subset[a,b]$ are arbitrarily given.

It is not assumed that the coefficients $A(\,\cdot\,,\varepsilon)$, $\beta_{j,k}^{(l)}(\varepsilon)$ or points $t_{j,k}(\varepsilon)$ have a certain regularity on the parameter ε as $\varepsilon>0$. It will be required that for each fixed $j\in\{1,\dots,r\}$ all the points $t_{j,k}(\varepsilon)$ have a common limit as $\varepsilon\to0+$, but for the zero-point series $t_{0,k}(\varepsilon)$ this requirement will not be necessary.

The solution $y=y(\,\cdot\,,\varepsilon)$ of the multi-point boundary-value problem (5), (9) is continuous on the parameter ε if it exists, is unique, and satisfies the limit relation

$$\big\|y(\,\cdot\,,\varepsilon)-y(\,\cdot\,,0)\big\|_{n,p}\longrightarrow0\ \text{ as }\ \varepsilon\to0+\,. \tag{10}$$

Consider the following assumptions as $\varepsilon\to0+$ and $p=\infty$:

(α) $t_{j,k}(\varepsilon)\to t_j$ for all $j\in\{1,\dots,r\}$, and $k\in\{1,\dots,\omega_j(\varepsilon)\}$;

(β) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \longrightarrow \beta_j^{(l)}$ for all $j \in \{1,\dots,r\}$, and $l \in \{0,\dots,n\}$;

(γ) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\|\,|t_{j,k}(\varepsilon)-t_j| \longrightarrow 0$ for all $j \in \{1,\dots,r\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n\}$;

(δ) $\displaystyle\sum_{k=1}^{\omega_0(\varepsilon)} \|\beta_{0,k}^{(l)}(\varepsilon)\| \longrightarrow$ for all $k \in \{1,\dots,\omega_0(\varepsilon)\}$, and $l \in \{0,\dots,n\}$.

Assumptions (β) and (γ) imply that the norms of the coefficients $\beta_{j,k}^{(l)}(\varepsilon)$ can increase as $\varepsilon \to 0+$, but not too fast.

Theorem 5. *Let the boundary-value problem* (5), (9) *for $p = \infty$ satisfy the assumptions* (α), (β), (γ), (δ). *Then it satisfies the limit condition* (II). *If, moreover, the conditions* (0) *and* (I) *are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation* (10).

Consider also the following assumptions as $\varepsilon \to 0+$ and $1 \leqslant p < \infty$:

(γ_p) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(n)}(\varepsilon)\|\,|t_{j,k}(\varepsilon)-t_j|^{1/p'} = O(1)$ for all $j \in \{1,\dots,r\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(γ') $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\|\,|t_{j,k}(\varepsilon)-t_j| \longrightarrow 0$ for all $j\in\{1,\dots,r\}$, $k\in\{1,\dots,\omega_j(\varepsilon)\}$, and $l\in\{0,\dots,n-1\}$.

Theorem 6. *Let the boundary-value problem* (5), (9) *for $1 \leqslant p < \infty$ satisfy the assumptions* (α), (β), (γ_p), (γ'), (δ). *Then it satisfies the limit condition* (II). *If, moreover, the conditions* (0) *and* (I) *are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation* (10).

The results are published in [1–4]. They allow extension for the systems of differential equations of higher order [5] and for boundary-value problems in Hölder spaces [6].

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