On Qualitative Properties of Minimizers for an Extremal Problem to Parabolic Equations

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1 Introduction

Consider the mixed boundary value problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ t > 0, \tag{1.2}$$

$$u(x,0) = \xi(x), \ \ 0 < x < 1, \tag{1.3}$$

where a and b are sufficiently smooth functions on $\overline{Q}_T$, $0 < a_0 \le a(x,t) \le a_1 < \infty$, $|b(x,t)| \le b_1 < \infty$, $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. We treat the functions ξ and ψ as fixed and the function φ as a control function to be found. The problem is to find a control function $\varphi = \varphi_0$ making the temperature $u(x,t)$ at some fixed point $x = c \in (0,1)$ maximally close to a given one, $z(t)$, during the whole time interval $(0,T)$. The quality of the control is estimated by the quadratic cost functional

$$J[z,\varphi] = \int_0^T (u_\varphi(c,t) - z(t))^2 \, dt, \tag{1.4}$$

where the function $u_\varphi(x,t)$ is a solution to problem (1.1)–(1.3). This problem arises while studying the problem of the temperature control in industrial greenhouses (see [6, 8]). Note that various extremum problems for partial differential equations with integral functionals were considered by different authors, a survey is contained in [12, 14], see also [6, 9].

The main difference between the problem considered in this paper and in previous works consists in the type of observation. We consider the pointwise observation contrary to the previously studied control problems with final and distributed observation (see, for example, [11]).

This paper develops results obtained in [2–4, 6–8]. We consider more general problem (the equation with variable coefficient $a = a(x,t)$, convection term and a non-homogeneous initial condition), and prove new results on qualitative properties of its minimizer. We prove these results by methods of qualitative theory of differential equations and, in particular, by some methods described in [1, 5].

2 Notations, definitions and preliminary results

Definition 2.1 (see [10, p. 26])**.** By $V_2^{1,0}(Q_T)$ we denote the Banach space of all functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

Definition 2.2. By $\widetilde{W}_2^1(Q_T)$ we denote the space of all functions $\eta \in W_2^1(Q_T)$ satisfying $\eta(x,T)=0$, $\eta(0,t)=0$.

Definition 2.3. We say that a function $u \in V_2^{1,0}(Q_T)$ is a weak solution to problem (1.1)–(1.3) if it satisfies the boundary condition $u(0,t) = \varphi(t)$ and the integral identity

$$\int\limits_{Q_T} \big(a(x,t)u_x\eta_x - b(x,t)u_x\eta - u\eta_t\big)\,dx\,dt = \int\limits_0^1 \xi(x)\eta(x,0)\,dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\,dt$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$.

Theorem 2.1 ([8])**.** *There exists a unique weak solution $u \in V_2^{1,0}(Q_T)$ to problem* (1.1)–(1.3) *and this solution satisfies the following inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1\Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\Big),$$

where the constant C_1 is independent of φ, ψ, and ξ.

Hereafter we denote by u_φ the unique solution to problem (1.1)–(1.3) with $\varphi,\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$, existing according to Theorem 2.1.

Suppose $z \in L_2(0,T)$. Let $\Phi \subset W_2^1(0,T)$ be a bounded closed convex set of control functions. For some $c \in (0,1)$ consider the functional $J[z,\varphi]$ defined by (1.4) and put

$$m[z,\Phi] = \inf_{\varphi\in\Phi} J[z,\varphi]. \tag{2.1}$$

Definition 2.4. We call problem (1.1)–(1.3), (2.1) *densely controllable* on $Z \subset L_2(0,T)$ by Φ if for any $z \in Z$ we have $m[z,\Phi] = 0$.

For a necessary condition of optimality we will consider also the adjoint to (1.1)–(1.3), (2.1) mixed problem for the inverse parabolic equation

$$p_t + (a(x,t)p_x)_x - (b(x,t)p)_x = \delta(x-c)\otimes(u_\varphi(c,t) - z(t)),\ \ (x,t)\in Q_T, \tag{2.2}$$

$$p(0,t) = 0,\quad a(1,t)p_x(1,t) - b(1,t)p(1,t) = 0,\ \ 0<t<T, \tag{2.3}$$

$$p(x,T) = 0,\ \ 0<x<1, \tag{2.4}$$

where u_φ is a solution of problem (1.1)–(1.3).

Definition 2.5. We say that a function $p \in V_2^{1,0}(Q_T)$ is a weak solution to problem (2.2)–(2.4) if it satisfies the boundary condition $p(0,t) = 0$ and the integral identity

$$\int_{Q_T} \big((a(x,t)p_x - b(x,t)p)\eta_x + p\eta_t\big)\, dx\, dt = -\int_0^T (u_{\varphi_0}(c,t) - z(t))\eta(c,t)\, dt$$

for any function $\eta \in W_2^1(Q_T)$ satisfying $\eta(0,t) = 0$ and $\eta(x,0) = 0$.

3 Main results

We denote by φ_0 minimizer of problem (1.1)–(1.3), (2.1), and $\Phi \subset W_2^1(0,T)$ is a bounded closed convex set.

Theorem 3.1. *For any $z \in L_2(0,T)$ there exists a unique function $\varphi_0 \in \Phi$ such that $m[z,\Phi] = J[z,\varphi_0]$.*

Theorem 3.2. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t, $m[z,\Phi] > 0$, and φ_0 is a minimizer. Then $\varphi_0 \in \partial\Phi$.*

Theorem 3.3. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t, and Φ_j, $j = 1,2$, Φ_j, $j = 1,2$ are bounded convex closed sets in $W_2^1(0,T)$ such that $\Phi_2 \subset \operatorname{Int}\Phi_1$, and $m[z,\Phi_1] > 0$. Then $m[z,\Phi_1] < m[z,\Phi_2]$.*

Theorem 3.4. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t. Then for any $z \in L_2(0,T)$ the equality $m[z, W_2^1(0,T)] = 0$ holds.*

Theorem 3.4 states dense controllability on $L_2(0,T)$ by $W_2^1(0,T)$. To prove this result we use the Titchmarsh convolution theorem [13, Theorem 7].

Theorem 3.5. *Let $\varphi_0 \in \Phi$ be a minimizer. Then for any $\varphi \in \Phi$ the following inequality holds:*

$$\int_0^T (u_{\varphi_0}(c,t) - z(t))(u_\varphi(c,t) - u_{\varphi_0}(c,t))\, dt \geq 0.$$

Theorem 3.6. *There exists a unique weak solution $p \in V_2^{1,0}(Q_T)$ to problem* (2.2)–(2.4) *and this solution satisfies the following inequality*

$$\|p\|_{V_2^{1,0}(Q_T)} \leq C_2\Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)} + \|z\|_{L_2(0,T)}\Big),$$

where the constant C_2 is independent of φ, ψ, ξ and z.

Theorem 3.7. *Let $\varphi_0 \in \Phi$ be a minimizer. Then for any $\varphi \in \Phi$ the following inequality holds:*

$$\int_0^T a(0,t)p_x(0,t)(\varphi(t) - \varphi_0(t))\, dt \leq 0,$$

where p is a weak solution of problem (2.2)–(2.4) *with $\varphi = \varphi_0$.*

Theorems 3.5 and 3.7 give us necessary conditions to minimizer.

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