On the Well-Posedness of the Cauchy Problem for Generalized Ordinary Linear Differential Systems

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For the linear system of generalized ordinary differential equations

$$dx = dA_0(t)\cdot x + df_0(t) \ \text{ for } \ t\in I \tag{1}$$

we consider the Cauchy problem

$$x(t_0)=c_0, \tag{2}$$

where $I\subset\mathbb{R}$ is an interval, $A_0\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $f_0\in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, $t_0\in I$, $c_0\in\mathbb{R}^n$.

We use the notations.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all $n\times m$-matrix-functions with bounded variation components on the closed interval $[a,b]$ from I.

$\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ is the sets of all $n\times m$-matrix-functions with bounded variation components on every closed interval $[a,b]$ from I.

By a solution of system (1) we understand a vector function $x\in \mathrm{BV}(I;\mathbb{R}^n)$ such that

$$x(t)=x(s)+\int\limits_s^t dA_0(\tau)\,x(\tau) \ \text{ for } \ s<t, \ \ s,t\in I,$$

where the integral is considered in the Kurzweil sense (see, [4]).

We present some results from [1] and [2].

Let x_0 be the unique solution of problem $(1),(2)$.

Along with the Cauchy problem $(1),(2)$ consider the sequence of the Cauchy problems

$$dx = dA_k(t)\cdot x + df_k(t), \tag{1_k}$$

$$x(t_k)=c_k, \tag{2_k}$$

$(k=1,2,\dots)$, where $A_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in BV_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $t_k\in I$ $(k=1,2,\dots)$ and $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$.

We give the conditions both for each from the two problems:

(a) The Cauchy problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\sup_{t\in I}\|x_k(t)-x_0(t)\|=0, \tag{3}$$

and

(b) The Cauchy problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\|x_k(t)-x_0(t)\|=0. \tag{4}$$

We assume that

$$\lim_{k\to+\infty} t_k = t_0.$$

For the formulation of theorems we use the notations.

- $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of matrix-function X at the point t; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;
- $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of the components of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$;
- If $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}$, then

$$\mathcal{B}(X,Y)(a) = O_{n\times m},$$

$$\mathcal{B}(X,Y)(t) = X(t)Y(t) - X(a)Y(a) - \int\limits_a^t dX(\tau)\cdot Y(\tau) \ \text{ for } \ t\in I,$$

where $a \in I$ is a fixed point.

Definition 1. We say that the sequence $(A_k, f_k; t_k)$ $(k = 1, 2, \dots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{5}$$

problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Theorem 1. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$ and the sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that the conditions*

$$\det\big(I_n + (-1)^j d_jA_0(t)\big) \neq 0 \ \textit{ for } \ t\in I, \ \ (-1)^j(t-t_0) < 0 \ \textit{ and for } \ t = t_0$$
$$\textit{if } \ j\in\{1,2\} \ \textit{ is such that } \ (-1)^j(t_k - t_0) > 0 \ \textit{ for every } \ k\in\{1,2,\dots\} \tag{6}$$

hold. Then the inclusion

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{7}$$

is true if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$ such that the conditions

$$\inf\Big\{|\det(H_0(t))| : \ t\in I\Big\} > 0$$

and

$$\limsup_{k\to+\infty} \bigvee_I (H_k + \mathcal{B}(H_k, A_k)) < +\infty$$

hold, and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t),$$

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k, A_k)(t) - \mathcal{B}(H_k, A_k)(t_k)\big) = \mathcal{B}(H_0, A_0)(t) - \mathcal{B}(H_0, A_0)(t_0)$$

and

$$\lim_{k\to+\infty}\big(\mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k)\big) = \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0)$$

hold uniformly on I.

Remark 1. In Theorem 1 without loss of generality we can assume that $H_0(t) \equiv I_n$, where I_n is the identity $n \times n$ matrix.

Theorem 1′. Let

$$\det\big(I_n + (-1)^j d_j A_k(t)\big) \neq 0 \ \text{ for } \ t \in [a,b] \ \ (j = 1,2; \ k = 0,1,\dots).$$

Then inclusion (7) holds if and only if the conditions

$$\lim_{k\to+\infty} X_k^{-1}(t) = X_0^{-1}(t)$$

and

$$\lim_{k\to+\infty} \big(\mathcal{B}(X_k^{-1}, f_k)(t) - \mathcal{B}(X_k^{-1}, f_k)(t_k)\big) = \mathcal{B}(X_0^{-1}, f_0)(t) - \mathcal{B}(X_0^{-1}, f_0)(t_0)$$

hold uniformly on $[a,b]$, where X_0 and X_k are fundamental matrices of the homogeneous systems corresponding to systems (1) and (1_k), respectively, for every $k \in \{1,2,\dots\}$.

We also consider the case when the condition

$$\lim_{k\to+\infty} c_{kj} = c_{0j} \ \text{ if } \ j \in \{1,2\} \ \text{ is such that } \ (-1)^j (t_k - t_0) \geq 0 \ \ (k = 0,1,\dots) \tag{5_j}$$

holds instead or along with (5), where

$$c_{kj} = c_k + (-1)^j \big(d_j A_k(t_k) c_k + d_j f_k(t_k)\big) \ \ (j = 1,2; \ k = 0,1,\dots). \tag{8}$$

Note that if

$$\lim_{k\to+\infty} d_j A_k(t_k) = d_j A_0(t_0) \ \text{ and } \ \lim_{k\to+\infty} d_j f_k(t_k) = d_j f_0(t_0)$$

for some $j \in \{1,2\}$, then condition (5_j) follows from (5).

Theorem 2. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions* (5), (6) *hold. Let, moreover, the sequences of matrix- and vector functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ and $f_k \in BV_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$ and bounded sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ be such that conditions* (5_j),

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \bigg\{\|A_{kj}(t) - A_{0j}(t)\| \bigg(1 + \bigg|\bigvee_{t_k}^{t}(A_k - A_0)\bigg|\bigg)\bigg\} = 0$$

and

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \bigg\{\|f_{kj}(t) - f_{0j}\| \bigg(1 + \bigg|\bigvee_{t_k}^{t}(A_k - A_0)\bigg|\bigg)\bigg\} = 0$$

hold if $j \in \{1,2\}$ is such that $(-1)^j (t_k - t_0) \geq 0$ for every $k \in \{1,2,\dots\}$, where c_{kj} $(k = 0,1,\dots)$ are defined by (8),

$$A_{kj}(t) \equiv (-1)^j (A_k(t) - A_k(t_k)) - d_j A_k(t_k) \ \ (j = 1,2; k = 0,1,\dots)$$

and

$$f_{kj}(t) \equiv (-1)^j (f_k(t) - f_k(t_k)) - d_j f_k(t_k) \ \ (j = 1,2; k = 0,1,\dots).$$

Then the Cauchy problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and condition (4) *holds.*

It is evident that if condition (3) holds, then condition (4) holds as well. But the inverse proposition is not true, in general.

We give the corresponding example, which is simple modification of the example given in [3].

Example 1. Let $I = [-1, 1]$, $n = 1$, α_k $(k = 1, 2, \dots)$ and β_k $(k = 1, 2, \dots)$ be an arbitrary increasing in $[-1, 0)$ and decreasing in $(0, 1]$, respectively, sequences such that

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 0 \ \text{ and } \ \lim_{k\to\infty} \gamma_k = \gamma_0 \in [0, 1),$$

where $\gamma_k = \alpha_k(\alpha_k - \beta_k)^{-1}$.

Let $t_k = t_0 = 0$ $(k = 1, 2, \dots)$, $c_k = \exp(\gamma_k - \gamma_0)c_0$ $(k = 1, 2, \dots)$, where c_0 is arbitrary, $f_k(t) = f_0(t) \equiv 0_n$ $(k = 1, 2, \dots)$,

$$A_k(t) = \begin{cases} 0 & \text{for } \ t \in [-1, \alpha_k[, \\ \dfrac{t - \alpha_k}{\beta_k - \alpha_k} & \text{for } \ t \in [\alpha_k, \beta_k], \\ 1 & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

It is not difficult to verify that the unique solution of the corresponding homogeneous initial problem has the form

$$x_k(t) = \begin{cases} c_k & \text{for } \ t \in [-1, \alpha_k[\,, \\ c_k \exp\big(t(\beta_k - \alpha_k)^{-1}\big) & \text{for } \ t \in [\alpha_k, \beta_k], \\ c_k \exp(1) & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

So, condition (4) holds, where

$$x_0(t) = \begin{cases} c_0 & \text{for } \ t \in [-1, 0[\,, \\ c_0 \exp(\gamma_0) & \text{for } \ t = 0, \\ c_0 \exp(1) & \text{for } \ t \in]0, 1], \end{cases}$$

but (3) does not hold uniformly on $[0, 1]$, because the function $x_0(t)$ is discontinuous at the point $t = 0$.

On the other hand, in the "limit" equation

$$dx = dA_0^*(t) \cdot x,$$

the function A_0^* is defined as

$$A_0^*(t) = \begin{cases} 0 & \text{for } \ t \in [-1, 0[\,, \\ \gamma_0 & \text{for } \ t = 0, \\ 1 & \text{for } \ t \in]0, 1], \end{cases}$$

and, therefore, the unique solution of the equation under the condition $x(0) = c_0(1 - \gamma_0)^{-1}$ has the form

$$x_0^*(t) = \begin{cases} c_0 & \text{for } \ t \in [-1, 0[\,, \\ c_0(1 - \gamma_0)^{-1} & \text{for } \ t = 0, \\ c_0(2 - \gamma_0)(1 - \gamma_0)^{-1} & \text{for } \ t \in]0, 1]. \end{cases}$$

It is evident that $x_0^* \neq x_0$.

On the other hand, x_0 is the solution of the initial problem

$$dx = dA_0(t)\cdot x, \ \ x(0) = c_0\exp(\gamma_0),$$

where

$$A_0(t) = \begin{cases} 0 & \text{for } \ t \in [-1, 0[\,, \\ 1 - \exp(-\gamma_0) & \text{for } \ t = 0, \\ \exp(1-\gamma_0) - \exp(-\gamma_0) & \text{for } \ t \in \,]0, 1]. \end{cases}$$

The obtained "anomaly" corresponds to the statement of Theorem 2, in particular to condition (4), where $H_k(t) \equiv I_n$ $(k = 1, 2, \dots)$, and

$$h_k(t) = \begin{cases} c_0 - c_k & \text{for } \ t \in [-1, \alpha_k[\,, \\ c_0(1-\gamma_k)^{-1} - c_k\exp\big(t(\beta_k-\alpha_k)^{-1}\big) & \text{for } \ t \in [\alpha_k, \beta_k], \\ c_0(2-\gamma_k)\,(1-\gamma_k)^{-1} - c_k\,\exp(1) & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

It is evident that the functions $x_k^*(t) = x_k(t)$ are solutions of the problem

$$dx = dA_k^*(t)\cdot x, \ \ x(0) = c_0(1-\gamma_k)^{-1}$$

for every natural k, where

$$A_k^*(t) = \begin{cases} 0 & \text{for } \ t \in [-1, \alpha_k[\,, \\ \gamma_k & \text{for } \ t \in [\alpha_k, \beta_k], \\ 1 & \text{for } \ t \in \,]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

So, due to the conditions $\lim\limits_{k\to+\infty} \gamma_k = \gamma_0$, we have

$$\lim_{k\to+\infty} \sup_{t\in I,\, t\neq t_k} \|A_k^*(t) - A_0^*(t)\| = 0.$$

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