On the Well-Posedness of the Cauchy Problem for High Order Ordinary Linear Differential Equations

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We consider the question on the well-posedness of the Cauchy problem

$$u^{(n)}=\sum_{l=1}^{n}p_l(t)u^{(l-1)}+p_0(t)\ \text{ for }\ t\in I, \tag{1}$$

$$u^{(i-1)}(t_0)=c_{i0}\ \ (i=1,\dots,n), \tag{2}$$

where $p_l\in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n)$, $t_0\in I$ and $c_{io}\in\mathbb{R}$ $(i=1,\dots,n)$, and I is an arbitrary interval from $\mathbb{R}$.

By $\mathrm{AC}(I;\mathbb{R})$ we denote the set of all absolutely continuous functions defined on I.

Let u_0 $(u^{(i-1)}\in\mathrm{AC}(I;\mathbb{R}),\ i=1,\dots,n)$ be the unique solution of the Cauchy problem (1), (2).

Along with problem (1), (2) we consider the sequence of problems

$$u^{(n)}=\sum_{l=1}^{n}p_{lk}(t)u^{(l-1)}+p_{0k}(t)\ \text{ for }\ t\in I, \tag{1_k}$$

$$u^{(i-1)}(t_k)=c_{ik}\ \ (i=1,\dots,n) \tag{2_k}$$

$(k=1,2,\dots)$, where $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $t_k\in I$ and $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$.

Let

$$\lim_{k\to+\infty}t_k=t_0. \tag{3}$$

Definition 1. We say that the sequence $(p_{lk},\dots,p_{nk},p_{0k};t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(p_1,\dots,p_n,p_0;t_0))$ if for every $c_{i0}\in\mathbb{R}$ $(i=1,\dots,n)$ and a sequence $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;$ $k=1,2,\dots)$, satisfying the condition

$$\lim_{k\to+\infty}c_{ik}=c_{i0}\ \ (i=1,\dots,n), \tag{4}$$

the condition

$$\lim_{k\to+\infty}u_k^{(i-1)}(t)=u_0^{(i-1)}(t)\ \ (i=1,\dots,n) \tag{5}$$

holds uniformly on I, where u_k is the unique solution of the Cauchy problem (1_k), (2_k) for any natural k.

Along with equations (1) and (1_k) $(k = 1, 2, \dots)$ we consider the corresponding homogeneous equations

$$u^{(n)} = \sum_{l=1}^{n} p_l(t) u^{(i-1)} \ \text{ for } \ t \in I \tag{1_0}$$

and

$$u^{(n)} = \sum_{l=1}^{n} p_{lk}(t) u^{(i-1)} \ \text{ for } \ t \in I \tag{1_{0k}}$$

$(k = 1, 2, \dots)$.

If the functions v_i $(i = 1, \dots, n)$ are such that $v_i^{(l-1)}$ $(i, l = 1, \dots, n)$ are absolutely continuous, then by $w_0(v_1, \dots, v_n)(t) = \det((v_i^{(l-1)}(t))_{i,l=1}^n)$ we denote so called Wronskiĭ's determinant, and by $w_{il}(v_1, \dots, v_n)(t)$ $(i, l = 1, \dots, n)$ we denote a cofactor of the il-element of $w_0(v_1, \dots, v_n)$.

Let u_l $(l = 1, \dots, n)$ and u_{lk} $(l = 1, \dots, n;\ k = 1, 2, \dots)$ be the fundamental systems of solutions of the homogeneous systems $(1)_0)$ and (2_{0k}) $(k = 1, 2, \dots)$, respectively.

Theorem 1. *Let $p_l \in L_{loc}(I; \mathbb{R})$ $(l = 0, \dots, n)$, $p_{lk} \in L_{loc}(I; \mathbb{R})$ $(l = 0, \dots, n;\ k = 1, 2, \dots)$, $t_k \in I$ $(k = 0, 1, \dots)$ and $c_{lk} \in \mathbb{R}$ $(l = 1, \dots, n;\ k = 0, 1, \dots)$ be such that conditions* (3), (4) *and*

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \bigg\{ \sum_{l=1}^{n} \bigg| \int_{t_k}^{t} (p_{lk}(\tau) - p_l(\tau))\, d\tau \bigg| \bigg(1 + \sum_{l=1}^{n} \bigg| \int_{t_k}^{t} |p_{lk}(\tau) - p_l(\tau)|\, d\tau \bigg| \bigg) \bigg\} = 0 \tag{6}$$

hold. Then

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \sum_{i=1}^{n} \big| u_k^{(i-1)}(t) - u_0^{(i-1)}(t) \big| = 0, \tag{7}$$

where u_k is the unique solution of the Cauchy problem $(1_k), (2_k)$ for any natural k.

Below we give some sufficient conditions, as well necessary and sufficient conditions guaranteeing the inclusion

$$((p_{lk}, \dots, p_{nk}, p_{0k}; t_k))_{k=1}^{+\infty} \in \mathcal{S}(p_1, \dots, p_n, p_0; t_0). \tag{8}$$

Theorem 2. *Let $p_l \in L(I; \mathbb{R})$ $(l = 0, \dots, n)$, $p_{lk} \in L(I; \mathbb{R})$ $(l = 0, \dots, n;\ k = 1, 2, \dots)$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that condition* (3) *holds. Then inclusion* (8) *holds if and only if there exists a sequence of functions h_{il}, $h_{ilk} \in \mathrm{AC}(I; \mathbb{R})$ $(i, l = 1, \dots, n;\ k = 0, 1, \dots)$ such that the conditions*

$$\inf \Big\{ \big| \det((h_{il}(t))_{i,l=1}^n) \big| :\ t \in I \Big\} > 0 \tag{9}$$

and

$$\limsup_{k \to +\infty} \sum_{i,l=1}^{n} \int_I \Big| h'_{ilk}(t) + h_{1\, l-1\, k}(t) \operatorname{sgn}(l-1) + h_{1nk}(t) p_l(t) \Big|\, dt < +\infty \tag{10}$$

hold, and the conditions

$$\lim_{k \to +\infty} h_{ilk}(t) = h_{il}(t) \ \ (i, l = 1, \dots, n) \tag{11}$$

and

$$\lim_{k \to +\infty} \int_{t_k}^{t} h_{ink}(\tau) p_{lk}(\tau)\, d\tau = \int_{t_0}^{t} h_{in}(\tau) p_l(\tau)\, d\tau \ \ (i = 1, \dots, n;\ l = 0, \dots, n)$$

hold uniformly on I.

Theorem 3. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that condition* (3) *holds. Then inclusion* (8) *holds if and only if the conditions*

$$\lim_{k\to+\infty} u_{lk}^{(i-1)}(t) = u_l^{(i-1)}(t) \quad (i,l=1,\dots,n)$$

and

$$\lim_{k\to+\infty} \int_{a_*}^{t} \frac{w_{in}(u_{1k},\dots,u_{nk})(\tau)}{w_0(u_{1k},\dots,u_{nk})(\tau)}\, p_{0k}(\tau)\,d\tau = \int_{a_*}^{t} \frac{w_{in}(u_1,\dots,u_n)(\tau)}{w_0(u_1,\dots,u_n)(\tau)}\, p_0(\tau)\,d\tau \quad (i=1,\dots,n) \tag{12}$$

hold uniformly on I.

Theorem 4. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$, $t_k \in I$ $(k=0,1,\dots)$ and $c_{lk} \in \mathbb{R}$ $(l=1,\dots,n;\ k=0,1,\dots)$ be such that the conditions* (3), (4) *and*

$$\limsup_{k\to+\infty} \int_I \|p_{lk}(t)\|dt < +\infty \quad (l=1,\dots,n)$$

hold, and the condition

$$\lim_{k\to+\infty} \int_{t_k}^{t} p_{lk}(\tau)\,d\tau = \int_{t_0}^{t} p_l(\tau)\,d\tau \quad (l=0,\dots,n)$$

holds uniformly on I. Then condition (5) *holds uniformly on I, where u_k is the unique solution of the Cauchy problem* $(1_k),(2_k)$ *for any natural k.*

Corollary 1. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (3), (4) *and* (10) *hold, and conditions* (11) *and*

$$\lim_{k\to+\infty} \int_{t_k}^{t} h_{ink}(\tau)p_{lk}(\tau)\,d\tau = \int_{t_0}^{t} p_l^*(\tau)\,d\tau \quad (i=1,\dots,n;\ l=0,\dots,n)$$

hold uniformly on I, where $p_l^* \in L(I;\mathbb{R})$ $(l=0,\dots,n)$; $h_{il}, h_{ilk} \in \mathrm{AC}(I;\mathbb{R})$ $(i,l=1,\dots,n;\ k=0,1,\dots)$. Then the inclusion

$$\big((p_{lk},\dots,p_{nk},p_{0k};t_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(p_1-p_1^*,\dots,p_n-p_n^*,p_0-p_0^*;t_0)$$

holds.

Remark 1. In Theorem 2 and Corollary 1, without loss of generality we can assume that $h_{ii}(t)\equiv 1$ and $h_{il}(t)\equiv 0$ $(i\neq l;\ i,l=1,\dots,n)$. So condition (9) is valid evidently.

Remark 2. If $n=2$ in Theorem 3, then condition (12) has the form

$$\lim_{k\to+\infty} \int_{a_*}^{t} \frac{u_{1k}'(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau = \int_{a_*}^{t} \frac{u_1'(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau,$$

$$\lim_{k\to+\infty} \int_{a_*}^{t} \frac{u_{1k}(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau = \int_{a_*}^{t} \frac{u_1(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau.$$

In the last equalities we can take u_{2k} instead of u_{1k} $(k=1,2,\dots)$, and u_2 instead of u_1.

For the proof we use the well-known concept. It is well-known that if the function u is a solution of problem (1), (2), then the vector-function $x = (x_i)_{i=1}^n$, $x_i = u^{(i-1)}$ $(i = 1, \dots, n)$, will be a solution of the Cauchy problem for the linear system of ordinary differential equations

$$\frac{dx}{dt} = P(t)\,x + q(t),$$
$$x(t_0) = c_0,$$

where the matrix- and vector-functions $P(t) = (p_{il}(t))_{i,l=1}^n$ and $q(t) = (q_i(t))_{i=1}^n$ are defined, respectively, by

$$p_{il}(t) \equiv 0, \;\; p_{i\,i+1} \equiv 1 \;\; (l \neq i+1;\; i = 1, \dots, n-1;\; l = 1, \dots, n),$$
$$p_{nl}(t) \equiv p_l(t) \;\; (l = 1, \dots, n);$$
$$q_i(t) \equiv 0 \;\; (i = 1, \dots, n-1), \;\; q_n(t) \equiv p_0(t),$$

and $c_0 = (c_{i0})_{i=1}^n$.

Analogously, problem $(1_k), (2_k)$ can be rewriten in the form of the last type problem for every natural k. So, using the results contained in [1] and [2], we get the results given above.

References

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