Finite Difference Approximation of Modified Burgers Equation in Sobolev Spaces

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We consider the initial boundary-value problem for the 1D cubic-nonlinear modified Burgers' equation with source term

$$\frac{\partial u}{\partial t} + (u)^2\,\frac{\partial u}{\partial x} - \mu\,\frac{\partial^2 u}{\partial x^2} = f, \ \ (x,t)\in Q := [0;1]\times[0;T], \tag{1}$$

$$u(0,t) = u(1,t) = 0, \ \ t\in[0,T), \quad u(x,0) = \varphi(x), \ \ x\in\Omega, \tag{2}$$

where $\Omega := [0;1]$, and the parameter $\mu = const > 0$.

A three-level finite difference scheme is constructed and investigated. Two-level scheme is used to find the values of unknown function on the first level. For each new level the obtained algebraic equations are linear with respect to the values of the unknown function.

Assume that a solution of this problem belongs to the fractional-order Sobolev spaces $W_2^k(Q)$, $k > 2$, whose norms and seminorms are denoted by a $\|\cdot\|_{W_2^k(Q)}$ and $|\cdot|_{W_2^k(Q)}$, respectively.

The finite domain Q is divided into rectangular grid by the points $(x_i,t_j) = (ih, j\tau)$, $i = 0,1,\dots,n$, $j = 0,1,2,\dots,J$, where $h = 1/n$ and $\tau = T/J$ denote the spatial and temporal mesh sizes, respectively.

Let

$$\overline{\omega} = \big\{x_i:\ i = 0,1,\dots,n\big\}, \ \ \omega = \big\{x_i:\ i = 1,2,\dots,n-1\big\}, \ \ \omega^+ = \big\{x_i:\ i = 1,2,\dots,n\big\}.$$

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is, $U(ih,j\tau) = U_i^j$. For the sake of simplicity sometimes we will use notations without subscripts: $U_i^j = U$, $U_i^{j+1} = \widehat{U}$, $U_i^{j-1} = \check{U}$. Moreover, let

$$\overline{U}^{\,0} = \frac{U^1 + U^0}{2}\,, \quad \overline{U}^{\,j} = \frac{U^{j+1} + U^{j-1}}{2}\,, \ \ j = 1,2,\dots\,.$$

We define the difference quotients in x and t directions as follows:

$$(U_i)_{\overline{x}} = \frac{U_i - U_{i-1}}{h}\,, \quad (U_i)_{\overset{\circ}{x}} = \frac{1}{2h}\,(U_{i+1} - U_{i-1}), \quad (U_i)_{\overline{x}x} = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\,,$$

$$(U)_t = \frac{\widehat{U} - \check{U}}{2\tau}\,, \ \ t = \tau, 2\tau, \dots, \quad (U^0)_t = \frac{U^1 - U^0}{\tau}\,.$$

Let H_0 be the set of functions defined on the mesh $\overline{\omega}$ and equal to zero at $x=0$ and $x=1$. On H_0 we define the following inner product and norm:

$$(U,V)=\sum_{x\in\omega} hU(x)V(x), \quad \|U\|=(U,U)^{1/2}.$$

Let, moreover,

$$(U,V]=\sum_{x\in\omega^+} hU(x)V(x), \quad \|U]|=(U,U]^{1/2}.$$

We need the following averaging operators for functions defined on Q:

$$\mathcal{S}v:=\frac{1}{\tau}\int\limits_0^{\tau} v(x,\zeta)\,d\zeta, \;\; t=0, \quad \mathcal{S}v:=\frac{1}{2\tau}\int\limits_{t-\tau}^{t+\tau} v(x,\zeta)\,d\zeta, \;\; t=\tau,2\tau,\dots,$$

$$\widehat{\mathcal{P}}v:=\frac{1}{h}\int\limits_x^{x+h} v(\xi,t)\,d\xi, \;\; x=0,h,\dots, \quad \mathcal{P}v:=\frac{1}{h^2}\int\limits_{x-h}^{x+h} (h-|x-\xi|)v(\xi,t)\,d\xi, \;\; x=h,2h,\dots.$$

Notice that

$$\mathcal{S}\,\frac{\partial v}{\partial t}=v_t, \quad \mathcal{P}\,\frac{\partial^2 v}{\partial x^2}=v_{\overline{x}\,x}.$$

We approximate problem (1), (2) with the help of the difference scheme:

$$\mathcal{L}U_i^j=F_i^j, \;\; i=1,2,\dots,n-1, \;\; j=0,1,\dots J-1, \tag{3}$$

$$U_0^j=U_n^j=0, \;\; j=0,1,\dots J, \quad U_i^0=\varphi(x_i), \;\; i=0,1,\dots,n, \tag{4}$$

where

$$F=\mathcal{P}f, \quad \mathcal{L}U:=U_t+\frac{1}{4}\,\Lambda U-\mu\overline{U}_{\overline{x}\,x}, \quad \Lambda U:=(U)^2\overline{U}_{\overset{\circ}{x}}+((U)^2\overline{U})_{\overset{\circ}{x}}.$$

Theorem 1. *The finite difference scheme* (3), (4) *is uniquely solvable.*

The proof of this theorem is based on partial summation formulas and the following identities

$$\big(YV_{\overset{\circ}{x}}+(YV)_{\overset{\circ}{x}},V\big)=0, \quad (V_{\overset{\circ}{x}},V)=0, \;\; \text{if} \;\; V\in H_0$$

as well.

Let $Z:=U-u$, where u is the exact solution of problem (1), (2), and U is the solution of the finite difference scheme (3), (4). Substituting $U=Z+u$ into (3), (4), we obtain the following problem for the error Z:

$$(Z^j)_t-\mu(Z^j)_{\overline{x}\,x}=-\frac{1}{4}\,(\Lambda U^j-\Lambda u^j)+\Psi^j, \;\; j=0,1,2,\dots, \tag{5}$$

$$Z^0=0, \;\; Z_0^j=Z_n^j=0, \;\; j=0,1,2,\dots \tag{6}$$

where

$$\Psi:=F-\mathcal{L}u.$$

Let

$$B^j:=\|Z^j\|^2+\|Z^{j-1}\|^2, \;\; j=1,2,\dots.$$

Lemma 1. *For a solution of problem* (5), (6) *the following relations are valid*

$$B^1 \leq \|\tau \Psi^0\|^2, \tag{7}$$

$$B^{j+1} \leq c_1 B^1 + c_2 \tau \sum_{k=1}^{j} \|\Psi^k\|^2, \ \ j = 1, 2, \dots . \tag{8}$$

In order to determine the rate of convergence of the finite difference scheme (3), (4) with the help of Lemma 1, it is sufficient to estimate the terms on the right-hand side of (7), (8). For this, we use a particular case of the Dupont–Scott approximation theorem [4] and it represents a generalization of Bramble–Hilbert lemma [3] (see, e.g. [1, 2, 5]).

Theorem 2. *Let the exact solution of the initial-boundary value problem* (1), (2) *belong to* $W_2^k(Q)$, $2 < k \leq 3$. *Then the convergence rate of the finite difference scheme* (3), (4) *is determined by the estimate*

$$\|U^j - u^j\| \leq c(\tau^{k-1} + h^{k-1}) \|u\|_{W_2^k(Q)},$$

where $c = c(u)$ *denotes a positive constant, independent of* h *and* τ.

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