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ABSTRACTS

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Finite Difference Approximation of Modified Burgers Equation in Sobolev Spaces

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We consider the initial boundary-value problem for the 1D cubic-nonlinear modified Burgers' equation with source term

$$\frac{\partial u}{\partial t}+(u)^2\,\frac{\partial u}{\partial x}-\mu\,\frac{\partial^2 u}{\partial x^2}=f,\ \ (x,t)\in Q:=[0;1]\times[0;T], \tag{1}$$

$$u(0,t)=u(1,t)=0,\ \ t\in[0,T),\quad u(x,0)=\varphi(x),\ \ x\in\Omega, \tag{2}$$

where $\Omega:=[0;1]$, and the parameter $\mu=const>0$.

A three-level finite difference scheme is constructed and investigated. Two-level scheme is used to find the values of unknown function on the first level. For each new level the obtained algebraic equations are linear with respect to the values of the unknown function.

Assume that a solution of this problem belongs to the fractional-order Sobolev spaces $W_2^k(Q)$, $k>2$, whose norms and seminorms are denoted by a $\|\cdot\|_{W_2^k(Q)}$ and $|\cdot|_{W_2^k(Q)}$, respectively.

The finite domain Q is divided into rectangular grid by the points $(x_i,t_j)=(ih,j\tau)$, $i=0,1,\dots,n$, $j=0,1,2,\dots,J$, where $h=1/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

Let

$$\overline{\omega}=\big\{x_i:\ i=0,1,\dots,n\big\},\ \ \omega=\big\{x_i:\ i=1,2,\dots,n-1\big\},\ \ \omega^+=\big\{x_i:\ i=1,2,\dots,n\big\}.$$

The value of mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is, $U(ih,j\tau)=U_i^j$. For the sake of simplicity sometimes we will use notations without subscripts: $U_i^j=U$, $U_i^{j+1}=\widehat{U}$, $U_i^{j-1}=\check{U}$. Moreover, let

$$\overline{U}^{\,0}=\frac{U^1+U^0}{2}\,,\quad \overline{U}^{\,j}=\frac{U^{j+1}+U^{j-1}}{2}\,,\ \ j=1,2,\dots\,.$$

We define the difference quotients in x and t directions as follows:

$$(U_i)_{\overline{x}}=\frac{U_i-U_{i-1}}{h}\,,\quad (U_i)_{\overset{\circ}{x}}=\frac{1}{2h}\,(U_{i+1}-U_{i-1}),\quad (U_i)_{\overline{x}x}=\frac{U_{i+1}-2U_i+U_{i-1}}{h^2}\,,$$

$$(U)_t=\frac{\widehat{U}-\check{U}}{2\tau}\,,\ \ t=\tau,2\tau,\dots,\quad (U^0)_t=\frac{U^1-U^0}{\tau}\,.$$

Let H_0 be the set of functions defined on the mesh $\overline{\omega}$ and equal to zero at $x=0$ and $x=1$. On H_0 we define the following inner product and norm:

$$(U,V)=\sum_{x\in\omega}hU(x)V(x),\quad \|U\|=(U,U)^{1/2}.$$

Let, moreover,

$$(U,V]=\sum_{x\in\omega^+}hU(x)V(x),\quad \|U]|=(U,U]^{1/2}.$$

We need the following averaging operators for functions defined on Q:

$$\mathcal{S}v:=\frac{1}{\tau}\int\limits_0^{\tau}v(x,\zeta)\,d\zeta,\ \ t=0,\quad \mathcal{S}v:=\frac{1}{2\tau}\int\limits_{t-\tau}^{t+\tau}v(x,\zeta)\,d\zeta,\ \ t=\tau,2\tau,\dots,$$

$$\widehat{\mathcal{P}}v:=\frac{1}{h}\int\limits_{x}^{x+h}v(\xi,t)\,d\xi,\ \ x=0,h,\dots,\quad \mathcal{P}v:=\frac{1}{h^2}\int\limits_{x-h}^{x+h}(h-|x-\xi|)v(\xi,t)\,d\xi,\ \ x=h,2h,\dots.$$

Notice that

$$\mathcal{S}\,\frac{\partial v}{\partial t}=v_t,\quad \mathcal{P}\,\frac{\partial^2 v}{\partial x^2}=v_{\overline{x}\,x}.$$

We approximate problem (1), (2) with the help of the difference scheme:

$$\mathcal{L}U_i^j=F_i^j,\ \ i=1,2,\dots,n-1,\ \ j=0,1,\dots J-1, \tag{3}$$

$$U_0^j=U_n^j=0,\ \ j=0,1,\dots J,\quad U_i^0=\varphi(x_i),\ \ i=0,1,\dots,n, \tag{4}$$

where

$$F=\mathcal{P}f,\quad \mathcal{L}U:=U_t+\frac{1}{4}\,\Lambda U-\mu\overline{U}_{\overline{x}\,x},\quad \Lambda U:=(U)^2\overline{U}_{\overset{\circ}{x}}+((U)^2\overline{U})_{\overset{\circ}{x}}.$$

Theorem 1. *The finite difference scheme* (3), (4) *is uniquely solvable.*

The proof of this theorem is based on partial summation formulas and the following identities

$$\big(YV_{\overset{\circ}{x}}+(YV)_{\overset{\circ}{x}},V\big)=0,\quad (V_{\overset{\circ}{x}},V)=0,\ \text{ if }\ V\in H_0$$

as well.

Let $Z:=U-u$, where u is the exact solution of problem (1), (2), and U is the solution of the finite difference scheme (3), (4). Substituting $U=Z+u$ into (3), (4), we obtain the following problem for the error Z:

$$(Z^j)_t-\mu(Z^j)_{\overline{x}\,x}=-\frac{1}{4}\,(\Lambda U^j-\Lambda u^j)+\Psi^j,\ \ j=0,1,2,\dots, \tag{5}$$

$$Z^0=0,\ \ Z_0^j=Z_n^j=0,\ \ j=0,1,2,\dots \tag{6}$$

where

$$\Psi:=F-\mathcal{L}u.$$

Let

$$B^j:=\|Z^j\|^2+\|Z^{j-1}\|^2,\ \ j=1,2,\dots.$$

Lemma 1. *For a solution of problem* $(5), (6)$ *the following relations are valid*

$$B^1 \le \|\tau \Psi^0\|^2, \tag{7}$$

$$B^{j+1} \le c_1 B^1 + c_2 \tau \sum_{k=1}^{j} \|\Psi^k\|^2, \ \ j = 1, 2, \dots . \tag{8}$$

In order to determine the rate of convergence of the finite difference scheme $(3), (4)$ with the help of Lemma 1, it is sufficient to estimate the terms on the right-hand side of (7), (8). For this, we use a particular case of the Dupont–Scott approximation theorem [4] and it represents a generalization of Bramble–Hilbert lemma [3] (see, e.g. [1, 2, 5]).

Theorem 2. *Let the exact solution of the initial-boundary value problem* $(1), (2)$ *belong to* $W_2^k(Q)$, $2 < k \le 3$. *Then the convergence rate of the finite difference scheme* $(3), (4)$ *is determined by the estimate*

$$\|U^j - u^j\| \le c(\tau^{k-1} + h^{k-1}) \|u\|_{W_2^k(Q)},$$

where $c = c(u)$ *denotes a positive constant, independent of* h *and* τ.

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On the Well-Posedness of the Cauchy Problem for High Order Ordinary Linear Differential Equations

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We consider the question on the well-posedness of the Cauchy problem

$$u^{(n)}=\sum_{l=1}^{n}p_l(t)u^{(l-1)}+p_0(t) \ \text{ for } \ t\in I, \tag{1}$$

$$u^{(i-1)}(t_0)=c_{i0} \ \ (i=1,\dots,n), \tag{2}$$

where $p_l\in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n)$, $t_0\in I$ and $c_{io}\in\mathbb{R}$ $(i=1,\dots,n)$, and I is an arbitrary interval from $\mathbb{R}$.

By $\mathrm{AC}(I;\mathbb{R})$ we denote the set of all absolutely continuous functions defined on I.

Let u_0 $(u^{(i-1)}\in\mathrm{AC}(I;\mathbb{R}),\ i=1,\dots,n)$ be the unique solution of the Cauchy problem (1), (2).

Along with problem (1), (2) we consider the sequence of problems

$$u^{(n)}=\sum_{l=1}^{n}p_{lk}(t)u^{(l-1)}+p_{0k}(t) \ \text{ for } \ t\in I, \tag{1_k}$$

$$u^{(i-1)}(t_k)=c_{ik} \ \ (i=1,\dots,n) \tag{2_k}$$

$(k=1,2,\dots)$, where $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $t_k\in I$ and $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$.

Let

$$\lim_{k\to+\infty}t_k=t_0. \tag{3}$$

Definition 1. We say that the sequence $(p_{lk},\dots,p_{nk},p_{0k};t_k)$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(p_1,\dots,p_n,p_0;t_0))$ if for every $c_{i0}\in\mathbb{R}$ $(i=1,\dots,n)$ and a sequence $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;$ $k=1,2,\dots)$, satisfying the condition

$$\lim_{k\to+\infty}c_{ik}=c_{i0} \ \ (i=1,\dots,n), \tag{4}$$

the condition

$$\lim_{k\to+\infty}u_k^{(i-1)}(t)=u_0^{(i-1)}(t) \ \ (i=1,\dots,n) \tag{5}$$

holds uniformly on I, where u_k is the unique solution of the Cauchy problem (1_k), (2_k) for any natural k.

Along with equations (1) and (1_k) $(k = 1, 2, \dots)$ we consider the corresponding homogeneous equations

$$u^{(n)} = \sum_{l=1}^{n} p_l(t) u^{(i-1)} \ \text{ for } \ t \in I \tag{1_0}$$

and

$$u^{(n)} = \sum_{l=1}^{n} p_{lk}(t) u^{(i-1)} \ \text{ for } \ t \in I \tag{1_{0k}}$$

$(k = 1, 2, \dots)$.

If the functions v_i $(i = 1, \dots, n)$ are such that $v_i^{(l-1)}$ $(i, l = 1, \dots, n)$ are absolutely continuous, then by $w_0(v_1, \dots, v_n)(t) = \det((v_i^{(l-1)}(t))_{i,l=1}^n)$ we denote so called Wronskiĭ's determinant, and by $w_{il}(v_1, \dots, v_n)(t)$ $(i, l = 1, \dots, n)$ we denote a cofactor of the il-element of $w_0(v_1, \dots, v_n)$.

Let u_l $(l = 1, \dots, n)$ and u_{lk} $(l = 1, \dots, n;\ k = 1, 2, \dots)$ be the fundamental systems of solutions of the homogeneous systems $(1)_0)$ and (2_{0k}) $(k = 1, 2, \dots)$, respectively.

Theorem 1. *Let $p_l \in L_{loc}(I; \mathbb{R})$ $(l = 0, \dots, n)$, $p_{lk} \in L_{loc}(I; \mathbb{R})$ $(l = 0, \dots, n;\ k = 1, 2, \dots)$, $t_k \in I$ $(k = 0, 1, \dots)$ and $c_{lk} \in \mathbb{R}$ $(l = 1, \dots, n;\ k = 0, 1, \dots)$ be such that conditions* (3), (4) *and*

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \bigg\{ \sum_{l=1}^{n} \bigg| \int_{t_k}^{t} (p_{lk}(\tau) - p_l(\tau))\, d\tau \bigg| \bigg(1 + \sum_{l=1}^{n} \bigg| \int_{t_k}^{t} |p_{lk}(\tau) - p_l(\tau)|\, d\tau \bigg| \bigg) \bigg\} = 0 \tag{6}$$

hold. Then

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \sum_{i=1}^{n} \big| u_k^{(i-1)}(t) - u_0^{(i-1)}(t) \big| = 0, \tag{7}$$

where u_k is the unique solution of the Cauchy problem $(1_k), (2_k)$ for any natural k.

Below we give some sufficient conditions, as well necessary and sufficient conditions guaranteeing the inclusion

$$((p_{lk}, \dots, p_{nk}, p_{0k}; t_k))_{k=1}^{+\infty} \in \mathcal{S}(p_1, \dots, p_n, p_0; t_0). \tag{8}$$

Theorem 2. *Let $p_l \in L(I; \mathbb{R})$ $(l = 0, \dots, n)$, $p_{lk} \in L(I; \mathbb{R})$ $(l = 0, \dots, n;\ k = 1, 2, \dots)$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that condition* (3) *holds. Then inclusion* (8) *holds if and only if there exists a sequence of functions h_{il}, $h_{ilk} \in \mathrm{AC}(I; \mathbb{R})$ $(i, l = 1, \dots, n;\ k = 0, 1, \dots)$ such that the conditions*

$$\inf \Big\{ \big| \det((h_{il}(t))_{i,l=1}^n) \big| :\ t \in I \Big\} > 0 \tag{9}$$

and

$$\limsup_{k \to +\infty} \sum_{i,l=1}^{n} \int_I \Big| h'_{ilk}(t) + h_{1\, l-1\, k}(t) \operatorname{sgn}(l-1) + h_{1nk}(t) p_l(t) \Big|\, dt < +\infty \tag{10}$$

hold, and the conditions

$$\lim_{k \to +\infty} h_{ilk}(t) = h_{il}(t) \ \ (i, l = 1, \dots, n) \tag{11}$$

and

$$\lim_{k \to +\infty} \int_{t_k}^{t} h_{ink}(\tau) p_{lk}(\tau)\, d\tau = \int_{t_0}^{t} h_{in}(\tau) p_l(\tau)\, d\tau \ \ (i = 1, \dots, n;\ l = 0, \dots, n)$$

hold uniformly on I.

Theorem 3. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that condition* (3) *holds. Then inclusion* (8) *holds if and only if the conditions*

$$\lim_{k\to+\infty} u_{lk}^{(i-1)}(t) = u_l^{(i-1)}(t) \ \ (i,l=1,\dots,n)$$

and

$$\lim_{k\to+\infty} \int\limits_{a_*}^{t} \frac{w_{in}(u_{1k},\dots,u_{nk})(\tau)}{w_0(u_{1k},\dots,u_{nk})(\tau)}\, p_{0k}(\tau)\,d\tau = \int\limits_{a_*}^{t} \frac{w_{in}(u_1,\dots,u_n)(\tau)}{w_0(u_1,\dots,u_n)(\tau)}\, p_0(\tau)\,d\tau \ \ (i=1,\dots,n) \tag{12}$$

hold uniformly on I.

Theorem 4. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L_{loc}(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$, $t_k \in I$ $(k=0,1,\dots)$ and $c_{lk} \in \mathbb{R}$ $(l=1,\dots,n;\ k=0,1,\dots)$ be such that the conditions* (3), (4) *and*

$$\limsup_{k\to+\infty} \int\limits_I \|p_{lk}(t)\|dt < +\infty \ \ (l=1,\dots,n)$$

hold, and the condition

$$\lim_{k\to+\infty} \int\limits_{t_k}^{t} p_{lk}(\tau)\,d\tau = \int\limits_{t_0}^{t} p_l(\tau)\,d\tau \ \ (l=0,\dots,n)$$

holds uniformly on I. Then condition (5) *holds uniformly on I, where u_k is the unique solution of the Cauchy problem* $(1_k), (2_k)$ *for any natural k.*

Corollary 1. *Let $p_l \in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk} \in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (3), (4) *and* (10) *hold, and conditions* (11) *and*

$$\lim_{k\to+\infty} \int\limits_{t_k}^{t} h_{ink}(\tau)p_{lk}(\tau)\,d\tau = \int\limits_{t_0}^{t} p_l^*(\tau)\,d\tau \ \ (i=1,\dots,n;\ l=0,\dots,n)$$

hold uniformly on I, where $p_l^* \in L(I;\mathbb{R})$ $(l=0,\dots,n)$; $h_{il}, h_{ilk} \in \mathrm{AC}(I;\mathbb{R})$ $(i,l=1,\dots,n;$ $k=0,1,\dots)$. Then the inclusion

$$\big((p_{lk},\dots,p_{nk},p_{0k};t_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(p_1-p_1^*,\dots,p_n-p_n^*,p_0-p_0^*;t_0)$$

holds.

Remark 1. In Theorem 2 and Corollary 1, without loss of generality we can assume that $h_{ii}(t)\equiv 1$ and $h_{il}(t)\equiv 0$ $(i\neq l;\ i,l=1,\dots,n)$. So condition (9) is valid evidently.

Remark 2. If $n=2$ in Theorem 3, then condition (12) has the form

$$\lim_{k\to+\infty} \int\limits_{a_*}^{t} \frac{u_{1k}'(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau = \int\limits_{a_*}^{t} \frac{u_1'(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau,$$

$$\lim_{k\to+\infty} \int\limits_{a_*}^{t} \frac{u_{1k}(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau = \int\limits_{a_*}^{t} \frac{u_1(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau.$$

In the last equalities we can take u_{2k} instead of u_{1k} $(k=1,2,\dots)$, and u_2 instead of u_1.

For the proof we use the well-known concept. It is well-known that if the function u is a solution of problem (1), (2), then the vector-function $x = (x_i)_{i=1}^n$, $x_i = u^{(i-1)}$ $(i = 1, \dots, n)$, will be a solution of the Cauchy problem for the linear system of ordinary differential equations

$$\frac{dx}{dt} = P(t)\,x + q(t),$$
$$x(t_0) = c_0,$$

where the matrix- and vector-functions $P(t) = (p_{il}(t))_{i,l=1}^n$ and $q(t) = (q_i(t))_{i=1}^n$ are defined, respectively, by

$$p_{il}(t) \equiv 0, \;\; p_{i\,i+1} \equiv 1 \;\; (l \neq i+1;\ i = 1, \dots, n-1;\ l = 1, \dots, n),$$
$$p_{nl}(t) \equiv p_l(t) \;\; (l = 1, \dots, n);$$
$$q_i(t) \equiv 0 \;\; (i = 1, \dots, n-1), \;\; q_n(t) \equiv p_0(t),$$

and $c_0 = (c_{i0})_{i=1}^n$.

Analogously, problem $(1_k), (2_k)$ can be rewriten in the form of the last type problem for every natural k. So, using the results contained in [1] and [2], we get the results given above.

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On the Well-Posedness of the Cauchy Problem for Generalized Ordinary Linear Differential Systems

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For the linear system of generalized ordinary differential equations

$$dx = dA_0(t)\cdot x + df_0(t) \ \text{ for } \ t\in I \tag{1}$$

we consider the Cauchy problem

$$x(t_0) = c_0, \tag{2}$$

where $I\subset\mathbb{R}$ is an interval, $A_0\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $f_0\in \mathrm{BV}_{loc}(I;\mathbb{R}^n)$, $t_0\in I$, $c_0\in\mathbb{R}^n$.

We use the notations.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all $n\times m$-matrix-functions with bounded variation components on the closed interval $[a,b]$ from I.

$\mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ is the sets of all $n\times m$-matrix-functions with bounded variation components on every closed interval $[a,b]$ from I.

By a solution of system (1) we understand a vector function $x\in \mathrm{BV}(I;\mathbb{R}^n)$ such that

$$x(t) = x(s) + \int\limits_s^t dA_0(\tau)\,x(\tau) \ \text{ for } \ s<t, \ \ s,t\in I,$$

where the integral is considered in the Kurzweil sense (see, [4]).

We present some results from [1] and [2].

Let x_0 be the unique solution of problem $(1),(2)$.

Along with the Cauchy problem $(1),(2)$ consider the sequence of the Cauchy problems

$$dx = dA_k(t)\cdot x + df_k(t), \tag{1_k}$$

$$x(t_k) = c_k, \tag{2_k}$$

$(k=1,2,\dots)$, where $A_k\in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $f_k\in BV_{loc}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$, $t_k\in I$ $(k=1,2,\dots)$ and $c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$.

We give the conditions both for each from the two problems:

(a) The Cauchy problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\sup_{t\in I}\|x_k(t)-x_0(t)\| = 0, \tag{3}$$

and

(b) The Cauchy problem $(1_k),(2_k)$ has a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty}\sup_{t\in I,\,t\neq t_k}\|x_k(t)-x_0(t)\| = 0. \tag{4}$$

We assume that

$$\lim_{k\to+\infty} t_k = t_0.$$

For the formulation of theorems we use the notations.

- $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of matrix-function X at the point t; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;
- $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of the components of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$;
- If $X \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ and $Y \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times m}$, then

$$\mathcal{B}(X,Y)(a) = O_{n\times m},$$

$$\mathcal{B}(X,Y)(t) = X(t)Y(t) - X(a)Y(a) - \int\limits_a^t dX(\tau)\cdot Y(\tau) \ \text{ for } \ t \in I,$$

where $a \in I$ is a fixed point.

Definition 1. We say that the sequence $(A_k, f_k; t_k)$ $(k = 1, 2, \dots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{5}$$

problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and condition (3) holds.

Theorem 1. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $t_0 \in I$ and the sequence of points $t_k \in I$ $(k = 1, 2, \dots)$ be such that the conditions*

$$\det\big(I_n + (-1)^j d_j A_0(t)\big) \neq \ 0 \ \textit{ for } \ t \in I, \ \ (-1)^j(t - t_0) < 0 \ \textit{ and for } \ t = t_0$$
$$\textit{if } \ j \in \{1,2\} \ \textit{ is such that } \ (-1)^j(t_k - t_0) > 0 \ \textit{ for every } \ k \in \{1, 2, \dots\} \tag{6}$$

hold. Then the inclusion

$$((A_k, f_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(A_0, f_0; t_0) \tag{7}$$

is true if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$ such that the conditions

$$\inf\Big\{|\det(H_0(t))| : \ t \in I\Big\} > 0$$

and

$$\limsup_{k\to+\infty} \bigvee_I (H_k + \mathcal{B}(H_k, A_k)) < +\infty$$

hold, and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t),$$

$$\lim_{k\to+\infty} \big(\mathcal{B}(H_k, A_k)(t) - \mathcal{B}(H_k, A_k)(t_k)\big) = \mathcal{B}(H_0, A_0)(t) - \mathcal{B}(H_0, A_0)(t_0)$$

and

$$\lim_{k\to+\infty} \big(\mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k)\big) = \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0)$$

hold uniformly on I.

Remark 1. In Theorem 1 without loss of generality we can assume that $H_0(t) \equiv I_n$, where I_n is the identity $n \times n$ matrix.

Theorem 1′. Let

$$\det\big(I_n + (-1)^j d_j A_k(t)\big) \neq 0 \ \text{ for } \ t \in [a,b] \ \ (j = 1,2; \ k = 0,1,\dots).$$

Then inclusion (7) holds if and only if the conditions

$$\lim_{k \to +\infty} X_k^{-1}(t) = X_0^{-1}(t)$$

and

$$\lim_{k \to +\infty} \big(\mathcal{B}(X_k^{-1}, f_k)(t) - \mathcal{B}(X_k^{-1}, f_k)(t_k)\big) = \mathcal{B}(X_0^{-1}, f_0)(t) - \mathcal{B}(X_0^{-1}, f_0)(t_0)$$

hold uniformly on $[a,b]$, where X_0 and X_k are fundamental matrices of the homogeneous systems corresponding to systems (1) and (1_k), respectively, for every $k \in \{1,2,\dots\}$.

We also consider the case when the condition

$$\lim_{k \to +\infty} c_{kj} = c_{0j} \ \text{ if } \ j \in \{1,2\} \ \text{ is such that } \ (-1)^j (t_k - t_0) \geq 0 \ \ (k = 0,1,\dots) \tag{5_j}$$

holds instead or along with (5), where

$$c_{kj} = c_k + (-1)^j \big(d_j A_k(t_k) c_k + d_j f_k(t_k)\big) \ \ (j = 1,2; \ k = 0,1,\dots). \tag{8}$$

Note that if

$$\lim_{k \to +\infty} d_j A_k(t_k) = d_j A_0(t_0) \ \text{ and } \ \lim_{k \to +\infty} d_j f_k(t_k) = d_j f_0(t_0)$$

for some $j \in \{1,2\}$, then condition (5_j) follows from (5).

Theorem 2. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in BV(I;\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_0 \in I$, and the sequence of points $t_k \in I$ $(k = 1,2,\dots)$ be such that conditions (5), (6) hold. Let, moreover, the sequences of matrix- and vector functions $A_k \in \mathrm{BV}_{loc}(I;\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ and $f_k \in BV_{loc}(I;\mathbb{R}^n)$ $(k = 1,2,\dots)$ and bounded sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1,2,\dots)$ be such that conditions (5_j),*

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \bigg\{ \|A_{kj}(t) - A_{0j}(t)\| \bigg(1 + \bigg|\bigvee_{t_k}^{t}(A_k - A_0)\bigg|\bigg)\bigg\} = 0$$

and

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \bigg\{ \|f_{kj}(t) - f_{0j}\| \bigg(1 + \bigg|\bigvee_{t_k}^{t}(A_k - A_0)\bigg|\bigg)\bigg\} = 0$$

hold if $j \in \{1,2\}$ is such that $(-1)^j (t_k - t_0) \geq 0$ for every $k \in \{1,2,\dots\}$, where c_{kj} $(k = 0,1,\dots)$ are defined by (8),

$$A_{kj}(t) \equiv (-1)^j (A_k(t) - A_k(t_k)) - d_j A_k(t_k) \ \ (j = 1,2; k = 0,1,\dots)$$

and

$$f_{kj}(t) \equiv (-1)^j (f_k(t) - f_k(t_k)) - d_j f_k(t_k) \ \ (j = 1,2; k = 0,1,\dots).$$

Then the Cauchy problem $(1_k), (2_k)$ has a unique solution x_k for any sufficiently large k and condition (4) holds.

It is evident that if condition (3) holds, then condition (4) holds as well. But the inverse proposition is not true, in general.

We give the corresponding example, which is simple modification of the example given in [3].

Example 1. Let $I=[-1,1]$, $n=1$, α_k $(k=1,2,\dots)$ and β_k $(k=1,2,\dots)$ be an arbitrary increasing in $[-1,0)$ and decreasing in $(0,1]$, respectively, sequences such that

$$\lim_{k\to\infty}\alpha_k=\lim_{k\to\infty}\beta_k=0 \ \text{ and } \ \lim_{k\to\infty}\gamma_k=\gamma_0\in[0,1),$$

where $\gamma_k=\alpha_k(\alpha_k-\beta_k)^{-1}$.

Let $t_k=t_0=0$ $(k=1,2,\dots)$, $c_k=\exp(\gamma_k-\gamma_0)c_0$ $(k=1,2,\dots)$, where c_0 is arbitrary, $f_k(t)=f_0(t)\equiv 0_n$ $(k=1,2,\dots)$,

$$A_k(t)=\begin{cases}0 & \text{for } \ t\in[-1,\alpha_k[,\\ \dfrac{t-\alpha_k}{\beta_k-\alpha_k} & \text{for } \ t\in[\alpha_k,\beta_k],\\ 1 & \text{for } \ t\in]\beta_k,1] \ \ (k=1,2,\dots).\end{cases}$$

It is not difficult to verify that the unique solution of the corresponding homogeneous initial problem has the form

$$x_k(t)=\begin{cases}c_k & \text{for } \ t\in[-1,\alpha_k[\,,\\ c_k\exp\big(t(\beta_k-\alpha_k)^{-1}\big) & \text{for } \ t\in[\alpha_k,\beta_k],\\ c_k\exp(1) & \text{for } \ t\in]\beta_k,1] \ \ (k=1,2,\dots).\end{cases}$$

So, condition (4) holds, where

$$x_0(t)=\begin{cases}c_0 & \text{for } \ t\in[-1,0[\,,\\ c_0\exp(\gamma_0) & \text{for } \ t=0,\\ c_0\exp(1) & \text{for } \ t\in]0,1],\end{cases}$$

but (3) does not hold uniformly on $[0,1]$, because the function $x_0(t)$ is discontinuous at the point $t=0$.

On the other hand, in the "limit" equation

$$dx=dA_0^*(t)\cdot x,$$

the function A_0^* is defined as

$$A_0^*(t)=\begin{cases}0 & \text{for } \ t\in[-1,0[\,,\\ \gamma_0 & \text{for } \ t=0,\\ 1 & \text{for } \ t\in]0,1],\end{cases}$$

and, therefore, the unique solution of the equation under the condition $x(0)=c_0(1-\gamma_0)^{-1}$ has the form

$$x_0^*(t)=\begin{cases}c_0 & \text{for } \ t\in[-1,0[\,,\\ c_0(1-\gamma_0)^{-1} & \text{for } \ t=0,\\ c_0(2-\gamma_0)(1-\gamma_0)^{-1} & \text{for } \ t\in]0,1].\end{cases}$$

It is evident that $x_0^*\neq x_0$.

On the other hand, x_0 is the solution of the initial problem

$$dx = dA_0(t) \cdot x, \ \ x(0) = c_0 \exp(\gamma_0),$$

where

$$A_0(t) = \begin{cases} 0 & \text{for } \ t \in [-1, 0[\,, \\ 1 - \exp(-\gamma_0) & \text{for } \ t = 0, \\ \exp(1 - \gamma_0) - \exp(-\gamma_0) & \text{for } \ t \in \,]0, 1]. \end{cases}$$

The obtained "anomaly" corresponds to the statement of Theorem 2, in particular to condition (4), where $H_k(t) \equiv I_n$ $(k = 1, 2, \dots)$, and

$$h_k(t) = \begin{cases} c_0 - c_k & \text{for } \ t \in [-1, \alpha_k[\,, \\ c_0(1 - \gamma_k)^{-1} - c_k \exp\big(t(\beta_k - \alpha_k)^{-1}\big) & \text{for } \ t \in [\alpha_k, \beta_k], \\ c_0(2 - \gamma_k)\,(1 - \gamma_k)^{-1} - c_k \, \exp(1) & \text{for } \ t \in]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

It is evident that the functions $x^*_k(t) = x_k(t)$ are solutions of the problem

$$dx = dA^*_k(t) \cdot x, \ \ x(0) = c_0(1 - \gamma_k)^{-1}$$

for every natural k, where

$$A^*_k(t) = \begin{cases} 0 & \text{for } \ t \in [-1, \alpha_k[\,, \\ \gamma_k & \text{for } \ t \in [\alpha_k, \beta_k], \\ 1 & \text{for } \ t \in \,]\beta_k, 1] \ \ (k = 1, 2, \dots). \end{cases}$$

So, due to the conditions $\lim\limits_{k \to +\infty} \gamma_k = \gamma_0$, we have

$$\lim_{k \to +\infty} \sup_{t \in I,\, t \neq t_k} \|A^*_k(t) - A^*_0(t)\| = 0.$$

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On Qualitative Properties of Minimizers for an Extremal Problem to Parabolic Equations

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1 Introduction

Consider the mixed boundary value problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ t > 0, \tag{1.2}$$

$$u(x,0) = \xi(x), \ \ 0 < x < 1, \tag{1.3}$$

where a and b are sufficiently smooth functions on $\overline{Q}_T$, $0 < a_0 \le a(x,t) \le a_1 < \infty$, $|b(x,t)| \le b_1 < \infty$, $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. We treat the functions ξ and ψ as fixed and the function φ as a control function to be found. The problem is to find a control function $\varphi = \varphi_0$ making the temperature $u(x,t)$ at some fixed point $x = c \in (0,1)$ maximally close to a given one, $z(t)$, during the whole time interval $(0,T)$. The quality of the control is estimated by the quadratic cost functional

$$J[z,\varphi] = \int_0^T (u_\varphi(c,t) - z(t))^2 \, dt, \tag{1.4}$$

where the function $u_\varphi(x,t)$ is a solution to problem (1.1)–(1.3). This problem arises while studying the problem of the temperature control in industrial greenhouses (see [6, 8]). Note that various extremum problems for partial differential equations with integral functionals were considered by different authors, a survey is contained in [12, 14], see also [6, 9].

The main difference between the problem considered in this paper and in previous works consists in the type of observation. We consider the pointwise observation contrary to the previously studied control problems with final and distributed observation (see, for example, [11]).

This paper develops results obtained in [2–4, 6–8]. We consider more general problem (the equation with variable coefficient $a = a(x,t)$, convection term and a non-homogeneous initial condition), and prove new results on qualitative properties of its minimizer. We prove these results by methods of qualitative theory of differential equations and, in particular, by some methods described in [1, 5].

2 Notations, definitions and preliminary results

Definition 2.1 (see [10, p. 26])**.** By $V_2^{1,0}(Q_T)$ we denote the Banach space of all functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

Definition 2.2. By $\widetilde{W}_2^1(Q_T)$ we denote the space of all functions $\eta \in W_2^1(Q_T)$ satisfying $\eta(x,T)=0$, $\eta(0,t)=0$.

Definition 2.3. We say that a function $u \in V_2^{1,0}(Q_T)$ is a weak solution to problem (1.1)–(1.3) if it satisfies the boundary condition $u(0,t) = \varphi(t)$ and the integral identity

$$\int\limits_{Q_T} \big(a(x,t)u_x\eta_x - b(x,t)u_x\eta - u\eta_t\big)\,dx\,dt = \int\limits_0^1 \xi(x)\eta(x,0)\,dx + \int\limits_0^T a(1,t)\psi(t)\,\eta(1,t)\,dt$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$.

Theorem 2.1 ([8])**.** *There exists a unique weak solution $u \in V_2^{1,0}(Q_T)$ to problem* (1.1)–(1.3) *and this solution satisfies the following inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1\Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\Big),$$

where the constant C_1 is independent of φ, ψ, and ξ.

Hereafter we denote by u_φ the unique solution to problem (1.1)–(1.3) with $\varphi, \psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$, existing according to Theorem 2.1.

Suppose $z \in L_2(0,T)$. Let $\Phi \subset W_2^1(0,T)$ be a bounded closed convex set of control functions. For some $c \in (0,1)$ consider the functional $J[z,\varphi]$ defined by (1.4) and put

$$m[z,\Phi] = \inf_{\varphi\in\Phi} J[z,\varphi]. \tag{2.1}$$

Definition 2.4. We call problem (1.1)–(1.3), (2.1) *densely controllable* on $Z \subset L_2(0,T)$ by Φ if for any $z \in Z$ we have $m[z,\Phi] = 0$.

For a necessary condition of optimality we will consider also the adjoint to (1.1)–(1.3), (2.1) mixed problem for the inverse parabolic equation

$$p_t + (a(x,t)p_x)_x - (b(x,t)p)_x = \delta(x-c) \otimes (u_\varphi(c,t) - z(t)), \;\; (x,t) \in Q_T, \tag{2.2}$$

$$p(0,t) = 0, \quad a(1,t)p_x(1,t) - b(1,t)p(1,t) = 0, \;\; 0 < t < T, \tag{2.3}$$

$$p(x,T) = 0, \;\; 0 < x < 1, \tag{2.4}$$

where u_φ is a solution of problem (1.1)–(1.3).

Definition 2.5. We say that a function $p \in V_2^{1,0}(Q_T)$ is a weak solution to problem (2.2)–(2.4) if it satisfies the boundary condition $p(0,t) = 0$ and the integral identity

$$\int\limits_{Q_T} \big((a(x,t)p_x - b(x,t)p)\eta_x + p\eta_t\big)\,dx\,dt = -\int\limits_0^T (u_{\varphi_0}(c,t) - z(t))\eta(c,t)\,dt$$

for any function $\eta \in W_2^1(Q_T)$ satisfying $\eta(0,t) = 0$ and $\eta(x,0) = 0$.

3 Main results

We denote by φ_0 minimizer of problem (1.1)–(1.3), (2.1), and $\Phi \subset W_2^1(0,T)$ is a bounded closed convex set.

Theorem 3.1. *For any $z \in L_2(0,T)$ there exists a unique function $\varphi_0 \in \Phi$ such that $m[z,\Phi] = J[z,\varphi_0]$.*

Theorem 3.2. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t, $m[z,\Phi] > 0$, and φ_0 is a minimizer. Then $\varphi_0 \in \partial\Phi$.*

Theorem 3.3. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t, and Φ_j, $j = 1,2$, Φ_j, $j = 1,2$ are bounded convex closed sets in $W_2^1(0,T)$ such that $\Phi_2 \subset \operatorname{Int}\Phi_1$, and $m[z,\Phi_1] > 0$. Then $m[z,\Phi_1] < m[z,\Phi_2]$.*

Theorem 3.4. *Suppose the coefficients a and b in equation* (1.1) *do not depend on t. Then for any $z \in L_2(0,T)$ the equality $m[z,W_2^1(0,T)] = 0$ holds.*

Theorem 3.4 states dense controllability on $L_2(0,T)$ by $W_2^1(0,T)$. To prove this result we use the Titchmarsh convolution theorem [13, Theorem 7].

Theorem 3.5. *Let $\varphi_0 \in \Phi$ be a minimizer. Then for any $\varphi \in \Phi$ the following inequality holds:*

$$\int\limits_0^T (u_{\varphi_0}(c,t) - z(t))(u_\varphi(c,t) - u_{\varphi_0}(c,t))\,dt \geq 0.$$

Theorem 3.6. *There exists a unique weak solution $p \in V_2^{1,0}(Q_T)$ to problem* (2.2)–(2.4) *and this solution satisfies the following inequality*

$$\|p\|_{V_2^{1,0}(Q_T)} \leq C_2\Big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)} + \|z\|_{L_2(0,T)}\Big),$$

where the constant C_2 is independent of φ, ψ, ξ and z.

Theorem 3.7. *Let $\varphi_0 \in \Phi$ be a minimizer. Then for any $\varphi \in \Phi$ the following inequality holds:*

$$\int\limits_0^T a(0,t)p_x(0,t)(\varphi(t) - \varphi_0(t))\,dt \leq 0,$$

where p is a weak solution of problem (2.2)–(2.4) *with $\varphi = \varphi_0$.*

Theorems 3.5 and 3.7 give us necessary conditions to minimizer.

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On Linear Boundary-Value Problems for Differential Systems in Sobolev spaces

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Let a finite interval $[a,b]\subset\mathbb{R}$ and parameters $\{m,n,r\}\subset\mathbb{N}$, $1\leqslant p\leqslant\infty$, be given. By $W_p^n=W_p^n([a,b];\mathbb{C}):=\{y\in C^{n-1}[a,b]:\ y^{(n-1)}\in AC[a,b],\ y^{(n)}\in L_p[a,b]\}$ we denote a complex Sobolev space and set $W_p^0:=L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n,p}=\sum_{k=0}^{n-1}\|y^{(k)}\|_p+\|y^{(n)}\|_p,$$

where $\|\cdot\|_p$ is the norm in the space $L_p([a,b];\mathbb{C})$. Similarly, by $(W_p^n)^m:=W_p^n([a,b];\mathbb{C}^m)$ and $(W_p^n)^{m\times m}:=W_p^n([a,b];\mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^n.

We consider the following linear boundary-value problem

$$Ly(t):=y'(t)+A(t)y(t)=f(t),\quad t\in(a,b), \tag{1}$$

$$By=c, \tag{2}$$

where the matrix-valued function $A(\cdot)\in(W_p^{n-1})^{m\times m}$, the vector-valued function $f(\cdot)\in(W_p^{n-1})^m$, the vector $c\in\mathbb{C}^r$, the linear continuous operator

$$B:(W_p^n)^m\to\mathbb{C}^r \tag{3}$$

are arbitrarily chosen; and the vector-valued function $y(\cdot)\in(W_p^n)^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution of the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot)\in(W_p^n)^m$ satisfying equation (1) almost everywhere on (a,b) (everywhere for $n\geq 2$) and equality (2) specifying r scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^n)^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^{n-1})^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multi-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives of the unknown functions of the order $k\leqslant n$.

It is known that, for $1\leq p<\infty$, every operator B in (3) admits a unique analytic representation

$$By=\sum_{k=0}^{n-1}\alpha_k y^{(k)}(a)+\int_a^b\Phi(t)y^{(n)}(t)\,\mathrm{d}t,\quad y(\cdot)\in(W_p^n)^m,$$

where the matrices $\alpha_k\in\mathbb{C}^{r\times m}$ and the matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{r\times m})$, $1/p+1/p'=1$.

For $p=\infty$ this formula also defines an operator $B\in L((W_\infty^n)^m;\mathbb{C}^r)$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

We rewrite the inhomogeneous boundary-value problem (1), (2) in the form of a linear operator equation $(L,B)y=(f,c)$, where (L,B) is a linear operator in the pair of Banach spaces

$$(L,B):(W_p^n)^m\to(W_p^{n-1})^m\times\mathbb{C}^r. \tag{4}$$

Recall that a linear continuous operator $T:X\to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind}T:=\dim\ker T-\dim(Y/T(X))$$

is finite.

Theorem 1. *The linear operator* (4) *is a bounded Fredholm operator with index $m-r$.*

Theorem 1 allows the next refinement.

By $Y(\,\cdot\,)\in(W_p^n)^{m\times m}$ we denote a unique solution of the linear homogenous matrix equation $(LY)(t)=O_m$, $Y(a)=I_m$, where O_m is the $(m\times m)$ zero matrix, and I_m is the $(m\times m)$ identity matrix.

Definition 1. A rectangular numerical matrix $M(L,B)\in\mathbb{C}^{m\times r}$ is characteristic for the boundary-value problem (1), (2) if its j-th column is the result of the action of the operator B on the j-th column of the matricant $Y(\,\cdot\,)$.

Here m is the number of scalar differential equations of the system (1), and r is the number of scalar boundary conditions.

Theorem 2. *The dimensions of the kernel and cokernel of the operator* (4) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix $M(L,B)$ respectively.*

Theorem 2 implies a criterion for the invertibility of the operator (4).

Corollary 1. *The operator (L,B) is invertible if and only if $r=m$ and the matrix $M(L,B)$ is nondegenerate.*

Let us consider parameterized by number $\varepsilon\in[0,\varepsilon_0)$, $\varepsilon_0>0$, linear boundary-value problem

$$L(\varepsilon)y(t;\varepsilon):=y'(t;\varepsilon)+A(t;\varepsilon)y(t;\varepsilon)=f(t;\varepsilon),\ \ t\in(a,b), \tag{5}$$

$$B(\varepsilon)y(\,\cdot\,;\varepsilon)=c(\varepsilon), \tag{6}$$

where for every fixed ε the matrix-valued function $A(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^{m\times m}$, the vector-valued function $f(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^m$, the vector $c(\varepsilon)\in\mathbb{C}^m$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon):(W_p^n)^m\to\mathbb{C}^m$, and the solution (the unknown vector-valued function) $y(\,\cdot\,;\varepsilon)\in(W_p^n)^m$.

It follows from Theorem 2 that the boundary-value problem (5), (6) is a Fredholm one with index zero.

Definition 2. A solution of the boundary-value problem (5), (6) continuously depends on the parameter ε for $\varepsilon=0$ if the following conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1<\varepsilon_0$ such that, for any $\varepsilon\in[0,\varepsilon_1)$ and arbitrary right-hand sides $f(\,\cdot\,;\varepsilon)\in(W_p^{n-1})^m$ and $c(\varepsilon)\in\mathbb{C}^m$ this problem has a unique solution $y(\,\cdot\,;\varepsilon)$ that belongs to the space $(W_p^n)^m$;

($**$) the convergence of the right-hand sides $f(\,\cdot\,;\varepsilon)\to f(\,\cdot\,;0)$ in $(W_p^{n-1})^m$ and $c(\varepsilon)\to c(0)$ in $\mathbb{C}^m$ as $\varepsilon\to0+$ implies the convergence of the solutions $y(\,\cdot\,;\varepsilon)\to y(\,\cdot\,;0)$ in $(W_p^n)^m$.

Consider the following conditions as $\varepsilon \to 0+$:

(0) limiting homogeneous boundary-value problem

$$L(0)y(t,0) = 0, \ \ t \in (a,b), \quad B(0)y(\,\cdot\,,0) = 0$$

has only the trivial solution;

(I) $A(\,\cdot\,,\varepsilon) \to A(\,\cdot\,,0)$ in the space $(W_p^{n-1})^{m\times m}$;

(II) $B(\varepsilon)y \to B(0)y$ in $\mathbb{C}^m$ for any $y \in (W_p^n)^m$.

Theorem 3. *A solution of the boundary-value problem* (5), (6) *continuously depends on the parameter ε for $\varepsilon = 0$ if and only if it satisfies condition (0) and the conditions* (I) *and* (II).

Consider the following quantities:

$$\big\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\big\|_{n,p}, \tag{7}$$

$$\widetilde{d}_{n-1,p}(\varepsilon) := \big\|L(\varepsilon)y(\,\cdot\,;0) - f(\,\cdot\,;\varepsilon)\big\|_{n-1,p} + \big\|B(\varepsilon)y(\,\cdot\,;0) - c(\varepsilon)\big\|_{\mathbb{C}^m}, \tag{8}$$

where (7) is the error and (8) is the discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ of the boundary-value problem (5), (6) if $y(\,\cdot\,;\varepsilon)$ is its exact solution and $y(\,\cdot\,;0)$ is an approximate solution of the problem.

Theorem 4. *Suppose that the boundary-value problem* (5), (6) *satisfies conditions (0), (I) and (II). Then there exist the positive quantities $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0, \varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\, \widetilde{d}_{n-1,p}(\varepsilon) \le \big\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\big\|_{n,p} \le \gamma_2\, \widetilde{d}_{n-1,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend of $y(\,\cdot\,;\varepsilon)$ and $y(\,\cdot\,;0)$.

According to this theorem, the error and discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ of the boundary-value problem (5), (6) have the same order of smallness.

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (5) multi-point Fredholm boundary condition

$$B(\varepsilon)y(\,\cdot\,,\varepsilon) = \sum_{j=0}^{r} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon), \varepsilon) = q(\varepsilon), \tag{9}$$

where the numbers $\{r, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^m$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m\times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a,b]$ are arbitrarily given.

It is not assumed that the coefficients $A(\,\cdot\,,\varepsilon)$, $\beta_{j,k}^{(l)}(\varepsilon)$ or points $t_{j,k}(\varepsilon)$ have a certain regularity on the parameter ε as $\varepsilon > 0$. It will be required that for each fixed $j \in \{1, \dots, r\}$ all the points $t_{j,k}(\varepsilon)$ have a common limit as $\varepsilon \to 0+$, but for the zero-point series $t_{0,k}(\varepsilon)$ this requirement will not be necessary.

The solution $y = y(\,\cdot\,,\varepsilon)$ of the multi-point boundary-value problem (5), (9) is continuous on the parameter ε if it exists, is unique, and satisfies the limit relation

$$\big\|y(\,\cdot\,,\varepsilon) - y(\,\cdot\,,0)\big\|_{n,p} \longrightarrow 0 \ \text{ as } \ \varepsilon \to 0+\,. \tag{10}$$

Consider the following assumptions as $\varepsilon \to 0+$ and $p = \infty$:

(α) $t_{j,k}(\varepsilon) \to t_j$ for all $j \in \{1, \dots, r\}$, and $k \in \{1, \dots, \omega_j(\varepsilon)\}$;

(β) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \longrightarrow \beta_j^{(l)}$ for all $j \in \{1,\dots,r\}$, and $l \in \{0,\dots,n\}$;

(γ) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\|\,|t_{j,k}(\varepsilon)-t_j| \longrightarrow 0$ for all $j \in \{1,\dots,r\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n\}$;

(δ) $\displaystyle\sum_{k=1}^{\omega_0(\varepsilon)} \|\beta_{0,k}^{(l)}(\varepsilon)\| \longrightarrow$ for all $k \in \{1,\dots,\omega_0(\varepsilon)\}$, and $l \in \{0,\dots,n\}$.

Assumptions (β) and (γ) imply that the norms of the coefficients $\beta_{j,k}^{(l)}(\varepsilon)$ can increase as $\varepsilon \to 0+$, but not too fast.

Theorem 5. *Let the boundary-value problem* (5), (9) *for* $p = \infty$ *satisfy the assumptions* (α), (β), (γ), (δ). *Then it satisfies the limit condition* (II). *If, moreover, the conditions* (0) *and* (I) *are fulfilled, then for a sufficiently small* ε *its solution exists, is unique and satisfies the limit relation* (10).

Consider also the following assumptions as $\varepsilon \to 0+$ and $1 \leqslant p < \infty$:

(γ_p) $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(n)}(\varepsilon)\|\,|t_{j,k}(\varepsilon) - t_j|^{1/p'} = O(1)$ for all $j \in \{1,\dots,r\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(γ') $\displaystyle\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\|\,|t_{j,k}(\varepsilon)-t_j| \longrightarrow 0$ for all $j \in \{1,\dots,r\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n-1\}$.

Theorem 6. *Let the boundary-value problem* (5), (9) *for* $1 \leqslant p < \infty$ *satisfy the assumptions* (α), (β), (γ_p), (γ'), (δ). *Then it satisfies the limit condition* (II). *If, moreover, the conditions* (0) *and* (I) *are fulfilled, then for a sufficiently small* ε *its solution exists, is unique and satisfies the limit relation* (10).

The results are published in [1–4]. They allow extension for the systems of differential equations of higher order [5] and for boundary-value problems in Hölder spaces [6].

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Generalization of Perron's and Vinograd's Examples of Lyapunov Exponents Instability to Linear Differential Systems with Parametric Perturbations

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For a given positive integer n let us denote by $\mathcal{M}_n$ the class of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

defined on the time semi-axis $\mathbb{R}_+$ with continuous bounded coefficients. Let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ denote the Lyapunov exponents [6, p. 561], [1, p. 38] of the system (1). Besides, we denote by $\mathcal{R}_n$ the subclass of the class $\mathcal{M}_n$ consisting of Lyapunov regular systems [6, p. 563], [1, p. 61]. In what follows, we identify the system (1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$ or $A \in \mathcal{R}_n$.

In the paper [7] O. Perron constructed an example of a system $A \in \mathcal{M}_2$ with negative Lyapunov exponents for which there exists an exponentially decaying perturbation $Q : \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ such that the largest Lyapunov exponent of the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}_+,$$

is positive. Put differently, the Lyapunov exponents, which are responsible for the stability, are not stable themselves (even under those perturbations of a system's coefficient matrix that decay exponentially).

As a result of Perron's example the problem naturally arises of finding wide enough subclasses of the class $\mathcal{M}_n$ consisting of the systems whose Lyapunov exponents are invariant under vanishing at infinity perturbations of the coefficient matrix. It was a long-standing conjecture that the class $\mathcal{R}_n$ of Lyapunov regular systems possesses the desired property. The conjecture was based essentially on the fundamental result by Lyapunov which claims that if a nonlinear system (with natural restrictions on the right-hand side) has a regular first approximation system and the latter is conditionally exponentially stable, then so is the zero solution of the original system (with the same dimension of the stable manifold and asymptotic exponent) [6, pp. 577–579]. Nevertheless, in the paper [8] R. E. Vinograd provided an example of a system $A \in \mathcal{R}_2$ whose Lyapunov exponents change under some vanishing at infinity perturbation of its coefficient matrix (the Lyapunov exponents of a regular system are invariant under exponentially decaying perturbations of its coefficient matrix, which is implied by Bogdanov–Grobman theorem [5, p. 188]).

Let M be a metric space. Let us introduce the classes $\mathcal{E}_n(M)$ and $\mathcal{Z}_n(M)$ of jointly continuous matrix-valued functions $Q(\,\cdot\,,\,\cdot\,)\colon \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$. The class $\mathcal{E}_n(M)$ consists of the functions $Q(\,\cdot\,,\,\cdot\,)$ exponentially decaying as $t \to +\infty$ with a uniform exponent with respect to $\mu \in M$:

$$\varlimsup_{t\to+\infty} t^{-1} \ln \|Q(t,\mu)\| < const < 0,$$

and the class $\mathcal{Z}_n(M)$ consists of the functions $Q(\,\cdot\,,\,\cdot\,)$ vanishing at infinity uniformly in $\mu \in M$:

$$\lim_{t\to+\infty} \sup_{\mu\in M} \|Q(t,\mu)\| = 0.$$

Generalizing the situation considered in examples of Perron and Vinograd, for each system $A \in \mathcal{M}_n$, let us define the class $\mathcal{P}_n(A;M)$ consisting of the families

$$\dot{x} = (A(t) + Q(t,\mu))x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{2}$$

of linear differential systems, where $\mu \in M$ is a parameter and $Q(\,\cdot\,,\,\cdot\,) \in \mathcal{E}_n(M)$. Next, for each $A \in \mathcal{R}_n$ we define the class $\mathcal{V}_n(A;M)$ to consist of those families (2) in which $Q(\,\cdot\,,\,\cdot\,) \in \mathcal{Z}_n(M)$. Therefore, fixing a value of the parameter $\mu \in M$ in the family (2) we obtain a linear differential system with continuous coefficients bounded on the semi-axis. Let $\lambda_1(\mu;A+Q) \leqslant \cdots \leqslant \lambda_n(\mu;A+Q)$ stand for the Lyapunov exponents of this system. Thus for each $k = \overline{1,n}$ we get the function $\lambda_k(\,\cdot\,;A)\colon M \to \mathbb{R}$, which is called the k-th Lyapunov exponent of the family (2), and the vector function $\Lambda(\,\cdot\,;A+Q) : M \to \mathbb{R}^n$ defined by $\Lambda(\mu;A+Q) = (\lambda_1(\mu;A+Q),\dots,\lambda_n(\mu;A+Q))$, which is called the spectrum of the Lyapunov exponents of the family (2).

We state the problems to be solved as follows: for each $n \in \mathbb{N}$ and every metric space M completely describe the classes of vector functions

$$\mathcal{P}_n(M) = \big\{\Lambda(\,\cdot\,;A+Q) \mid \; A \in \mathcal{M}_n, \; Q \in \mathcal{E}_n(M)\big\},$$
$$\mathcal{V}_n(M) = \big\{\Lambda(\,\cdot\,;A+Q) \mid \; A \in \mathcal{R}_n, \; Q \in \mathcal{Z}_n(M)\big\}.$$

Solutions to these problems will contain as special cases examples of Perron and Vinograd, respectively. If $n = 1$, then the descriptions of the above classes immediately follow from the definition of the Lyapunov exponent – for any metric space M both the classes $\mathcal{P}_1(M)$ and $\mathcal{V}_1(M)$ coincide with the class of constant functions $M \to \mathbb{R}$. Therefore, from now on, we assume that $n \geqslant 2$.

Let a vector function $f(\,\cdot\,) = (f_1(\,\cdot\,),\dots,f_n(\,\cdot\,)) : M \to \mathbb{R}^n$ belong to the class $\mathcal{P}_n(M)$ or to the class $\mathcal{V}_n(M)$. Let us state three properties of the vector function $f(\,\cdot\,)$ that it must satisfy (below these properties are numbered as 1), 2), 3)). One of the properties is trivially implied by the very definition of this vector function: 1) for every $\mu \in M$ the inequalities $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ hold. Another property follows from the fact that a matrix-valued function A is bounded on the time semi-axis and for every $\mu \in M$, a perturbation matrix $Q(\,\cdot\,,\mu)$ vanishes at infinity: 2) the vector function $f(\,\cdot\,)$ is bounded on M. For example, $|\Lambda(\mu;A+Q)| \leqslant n \sup\{\|A(t)\| \mid \; t \in \mathbb{R}_+\}$ for all $\mu \in M$. Before stating the third property let us recall that a function $g : M \to \mathbb{R}$ is said [4, p. 267] to be of the class $(^*,G_\delta)$ if for each $r \in \mathbb{R}$ the preimage $g^{-1}([r,+\infty))$ of the half-interval $[r,+\infty)$ is a G_δ-set of the metric space M. As follows from the paper [2], in which a complete description is obtained for the spectra of the Lyapunov exponents of general parametric families of linear differential systems continuous in the parameter uniformly in $t \in \mathbb{R}_+$, the property 3) is true: the components $f_k(\,\cdot\,)$ of the vector function $f(\,\cdot\,)$ are of the class $(^*,G_\delta)$.

Theorem 1. *Let M be a metric space, $n \geqslant 2$ an integer, and a vector function $f\colon M \to \mathbb{R}^n$ satisfy the properties* 1)–3). *Then there exist a system $A \in \mathcal{M}_n$ and a matrix-valued function $Q \in \mathcal{E}_n(M)$ such that the spectrum of the Lyapunov exponents of the family* (2) *coincides with the function f, i.e. $\Lambda(\mu;A+Q) = f(\mu)$ for all $\mu \in M$.*

Theorem 2. *Let M be a metric space, $n \geqslant 2$ an integer, and a vector function $f : M \to \mathbb{R}^n$ satisfy the properties* 1)–3). *Then there exist a Lyapunov regular system $A \in \mathcal{R}_n$ and a matrix-valued function $Q \in \mathcal{Z}_n(M)$ such that the spectrum of the Lyapunov exponents of the family* (2) *coincides with the function f, i.e. $\Lambda(\mu;A+Q) = f(\mu)$ for all $\mu \in M$.*

Thus, from the said above it follows that the classes $\mathcal{P}_n(M)$ and $\mathcal{V}_n(M)$ are identical, and their common complete description is contained in the following

Theorem 3. *For any $n \geqslant 2$ and every metric space M, a vector function $(f_1, \dots, f_n) : M \to \mathbb{R}^n$ belongs to the class $\mathcal{P}_n(M)$ (to the class $\mathcal{V}_n(M)$) if and only if it satisfies the properties* 1)–3)*. For each metric space M the class $\mathcal{P}_1(M)$ (the class $\mathcal{V}_1(M)$ coincides with the class of constant functions $M \to \mathbb{R}$.*

Note that if M is a segment of the real line, then in Theorems 1–3 above one can choose a matrix-valued function $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ to be analytical in $\mu \in M$ for each $t \in \mathbb{R}_+$.

Recall that a subset of a metric space M is said to be an $F_{\sigma\delta}$-set if it can be expressed as countable intersection of F_σ sets in M. The latter, in turn, are those which can be represented as countable unions of closed sets in M [4, p. 96]. Combining Theorem 2 above with [3, Corollary 2] we arrive at the following

Corollary. *Let an integer $n \geqslant 2$ and a metric space M be given. Then for any $F_{\sigma\delta}$-set S in M there exist a Lyapunov regular system $A \in \mathcal{R}_n$ and a matrix-valued function $Q \in \mathcal{Z}_n(M)$ such that S is the set of those $\mu \in M$ for which the system* (2) *is Lyapunov regular.*

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Asymptotic Representations of Solutions of Second Order Differential Equations with Nonlinearities, that are in Some Sense Near to Regularly Varying Functions

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')f(y,y'). \tag{1}$$

Here $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ are continuous functions, $f : \Delta_{Y_0} \times \Delta_{Y_1} \to]0,+\infty[$ is a continuously differentiable function, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} is a one-sided neighborhood of Y_i, $i \in \{0,1\}$. We suppose also that every function $\varphi_i(z)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0+\sigma_1 \ne 1$ and the function f satisfies the condition

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(v_0,v_1)}{f(v_0,v_1)} = 0 \ \text{ uniformly in } \ v_j \in \Delta_{Y_j}, \ \ j \ne k, \ \ k,j \in \{0,1\}. \tag{2}$$

Many works (see, e.g., [3,4,6]) have been devoted to the establishing of asymptotic representation of solutions of equations of the form (1) in case $f \equiv 1$. In the work, the right part of (1) was or in explicit form or asymptotically represented as the product of expressions, each of which depends only of t, or only of y, or only of y'. The fact is of the most importance. In general case equation (1) can contain nonlinearities of another types, for example, $e^{|\gamma \ln|y| + \mu \ln|y'||^{\alpha}}$, $0<\alpha<1$, $\gamma,\mu \in \mathbb{R}$.

Definition. The solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution if it is defined on $[t_0,\omega[\subset [a,\omega[$ and for all $i \in \{0,1\}$

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i, \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$.

We need the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t-\omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \ \ i = \in \{0,1\},$$

$$J_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \operatorname{sign} y_1^0 \int_{B_\omega^1}^{t} |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau,$$

$$B_\omega^1 = \begin{cases} b & \text{if } \ \displaystyle\int_b^\omega |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau = +\infty, \\ \omega & \text{if } \ \displaystyle\int_b^\omega |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau < +\infty, \end{cases}$$

$$I_1(t)=\alpha_0\Big|\frac{\lambda_0-1}{\lambda_0}\Big|^{\sigma_0}\int\limits_{A_\omega^1}^{t}\frac{(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau,\quad A_\omega^1=\begin{cases} a & \text{if } \displaystyle\int\limits_a^\omega \frac{p(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_a^\omega \frac{p(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau<+\infty,\end{cases}$$

$$J_2(t)=|\sigma_0|^{-\frac{1}{\sigma_0}}\operatorname{sign}y_1^0\int\limits_{B_\omega^2}^{t}|I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau,\quad B_\omega^2=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega |I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega |I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau<+\infty.\end{cases}$$

Theorem 1. *Let $\sigma_1\neq 1$. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, where $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, the next conditions are necessary*

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0,\quad \pi_\omega(t)\alpha_0y_1^0(\lambda_0-1)>0\ \text{ as } t\in[a,\omega[\,, \tag{3}$$

$$\lim_{t\uparrow\omega}y_0^0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}y_1^0|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_1'(t)}{J_1(t)}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\cdot\frac{\lambda_0}{\lambda_0-1}\,. \tag{4}$$

If

$$\lambda_0\neq\sigma_1-1\ \text{ or }\ (\sigma_1-1)(\sigma_0+\sigma_1-1)>0, \tag{5}$$

conditions (3), (4) *are sufficient for the existence of such solutions of equation* (1)*.*

For $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) *the next asymptotic representations take place as $t\uparrow\omega$,*

$$\begin{gathered}\frac{y(t)|y(t)|^{-\frac{\sigma_0}{1-\sigma_1}}}{(f(y(t),y'(t))\Theta_0(y(t))\Theta_1(y'(t)))^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,J_1(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)].\end{gathered} \tag{6}$$

By conditions (3), (5) and the first of the asymptotic representations (6), obtained in Theorem 1, it is clear that the case $\sigma_1=1$ requires a separate investigation. The following theorem covers this case.

Theorem 2. *Let $\sigma_1=1$. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, where $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, the next conditions are necessary and sufficient*

$$\begin{gathered}y_0^0J_2(t)>0,\ \ \alpha_0y_0^0\lambda_0>0,\ \ y_1^0\sigma_0I_1(t)<0\ \text{ as } t\in[a,\omega[\,,\\ \lim_{t\uparrow\omega}y_1^0|I_1(t)|^{-\frac{1}{\sigma_0}}=Y_1,\quad \lim_{t\uparrow\omega}y_0^0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\frac{\sigma_0}{1-\lambda_0}\,.\end{gathered}$$

For $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) *the next asymptotic representations take place as $t\uparrow\omega$,*

$$\begin{gathered}y(t)|\Theta_0(y(t))\Theta_1(y'(t))f(y(t),y'(t))|^{\frac{1}{\sigma_0}}=J_2(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)].\end{gathered}$$

For the equations of the form (1) the existence of different types of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions essentially depends from the orders σ_0 and σ_1 of the regularly varying functions φ_0, φ_1 as their arguments tend to Y_0, Y_1 respectively, and from the type of function p, that as must be mentioned, does not necessary have to be a regularly varying. By the results of Theorem 1, precisely by the third condition of (3), it is clear that $P_\omega(Y_0, Y_1, \lambda_0)$-solutions for which $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ may appear in the equations of the form (1), when p is regularly varying function as $t \uparrow \omega$. To simplify the calculations, we take $p(t) \equiv t^\gamma$. On the interval $[t_0; +\infty[$ $(t_0 > 0)$ we consider the differential equation

$$y'' = t^\gamma \varphi_0(y)\varphi_1(y') \exp\big(|\ln|y||^{\mu_0} + |\ln|y'||^{\mu_1}\big)^{\mu_2}, \tag{7}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$, $\mu_i \in (0, 1)$ for each $i \in \{0, 1, 2\}$. This equation is of the form (1), with $\alpha_0 = 1$, $p(t) = t^\gamma$, $f(y, y') = \exp(|\ln|y||^{\mu_0} + |\ln|y'||^{\mu_1})^{\mu_2}$. Now

$$\Delta_{Y_k} = [y_k^0, +\infty[\ \ (\forall\, k \in \{0, 1\}), \quad \omega = Y_0 = Y_1 = +\infty,$$

$$J_1(t) = \frac{1-\sigma_1}{\gamma - \sigma_1 + 2}\, |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \operatorname{sign} y_1^0 t^{\frac{\gamma-\sigma_1+2}{1-\sigma_1}}, \quad I_1(t) = \Big|\frac{\lambda_0 - 1}{\lambda_0}\Big|^{\sigma_0} \frac{t^{\sigma_0+\gamma+1}}{\sigma_0 + \gamma + 1},$$

$$J_2(t) = -\Big(\frac{\sigma_0 + \gamma + 1}{|\sigma_0|}\Big)^{\frac{1}{\sigma_0}} \frac{\sigma_0}{\gamma + 1} \operatorname{sign} y_1^0 t^{-\frac{\gamma+1}{\sigma_0}}.$$

Condition (2) in our case takes the following form

$$\lim_{\substack{v_k \to Y_k\\ v_k \in \Delta_{Y_k}}} \mu_k \mu_2 |\ln|v_k||^{\mu_k - 1}\big(|\ln|v_0||^{\mu_0} + |\ln|v_1||^{\mu_1}\big)^{\mu_2 - 1} = 0,$$

where $k \in \{0, 1\}$.

It is clear that since $m_i - 1 < 0$ for all $i \in \{0, 1, 2\}$, the function under the sign of a limit tends to zero uniformly over $v_j \in [y_k^0; +\infty[$, $j \neq k$, $k, j \in \{0, 1\}$.

We apply Theorem 1 and obtain that from all $P_{+\infty}(+\infty, +\infty, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, equation (7) can have only $P_{+\infty}(+\infty, +\infty, \frac{\gamma-\sigma_1+2}{\gamma+\sigma_1+1})$-solutions if

$$(\gamma - \sigma_1 + 2)(1 - \sigma_0 - \sigma_1) > 0, \quad \frac{1 - \sigma_0 - \sigma_1}{\gamma + \sigma_0 + 1} > 0.$$

These conditions are necessary, and if, together to them,

$$\frac{\gamma - \sigma_1 + 2}{\gamma + \sigma_0 + 1} \neq \sigma_1 + 1 \ \text{ or } \ (\sigma_1 - 1)(\sigma_0 + \sigma_1 - 1) > 0,$$

they are sufficient for the existence of such solutions of equation (7). In addition, for each such $P_{+\infty}(+\infty, +\infty, \frac{\gamma-\sigma_1+2}{\gamma+\sigma_1+1})$-solution of equation (7) the following asymptotic representations take place as $t \to +\infty$,

$$\frac{(y(t))^{1-\sigma_1}|y'(t)|^{\sigma_1}}{\varphi_0(y(t))\varphi_1(y'(t)) \exp(|\ln|y(t)||^{\mu_0} + |\ln|y'(t)||^{\mu_1})^{\mu_2}} = \Big(\frac{1 - \sigma_0 - \sigma_1}{\gamma - \sigma_1 + 2}\Big)^{1-\sigma_1} \Big|\frac{1 - \sigma_0 - \sigma_1}{\gamma + \sigma_0 + 1}\Big| t^{\gamma - \sigma_1 + 2}[1 + o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\gamma - \sigma_1 + 2}{1 - \sigma_0 - \sigma_1} \cdot \frac{1}{t}\, [1 + o(1)].$$

Then we also consider the differential equation (7) under the assumption that $\sigma_1 = 1$. We apply Theorem 2 and find that in this case from the $P_{+\infty}(+\infty, +\infty, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, equation (7) can have only $P_{+\infty}(+\infty, +\infty, \frac{\gamma+1}{\sigma_0+\gamma+1})$-solutions if

$$y_1^0 \sigma_0 < 0, \ \ y_0^0 \frac{\gamma + 1}{\sigma_0 + \gamma + 1} > 0.$$

This condition is necessary and sufficient for the existence of such solutions of equation (7). In addition, for any $P_{+\infty}(+\infty, +\infty, \frac{\gamma+1}{\sigma_0+\gamma+1})$-solution of equation (7) the following asymptotic representations take place as $t \to +\infty$,

$$y(t) \exp\Big(\frac{(|\ln|y||^{\mu_0} + |\ln|y'||^{\mu_1})^{\mu_2}}{\sigma_0}\Big) = -\Big(\frac{\sigma_0+\gamma+1}{|\sigma_0|}\Big)^{\frac{1}{\sigma_0}} \frac{\sigma_0}{\gamma+1} \operatorname{sign} y_1^0 t^{-\frac{\gamma+1}{\sigma_0}} [1+o(1)],$$

$$\frac{y'(t)}{y(t)} = -\frac{\gamma+1}{\sigma_0 t} [1+o(1)].$$

Another classes of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) have also been investigated before (see, e.g., [5]). The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equations like (1) has been considered only for cases when $f(y, y') \equiv 1$ and the function $\varphi_0(z)|z|^{-\sigma_0}$ satisfies some additional conditions. Later it has appeared an opportunity to extend the results onto more general cases (see, e.g., [1]). But functions that contain in their left side the derivative of the unknown function as it is in general case of equation (1), haven't been considered before. Let us notice that the derivative of every $P_\omega(Y_0, Y_1, \pm\infty)$-solution is a slowly varying function as $t \uparrow \omega$. It makes a lot of difficulties for the investigations. The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1) is established (see, [2]) for the case $f(y, y') \equiv \exp(R(|\ln|yy'||))$, $R :]0; +\infty[\to]0; +\infty[$ is continuously differentiable function, that is regularly varying on infinity of the order μ, $0 < \mu < 1$ and has monotone derivative.

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On the Solvability of Focal Boundary Value Problems for Higher-Order Linear Functional Differential Equations

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We obtain sharp solvability conditions for focal boundary value problems for higher-order linear functional differential equations with functional operators under integral and point-wise restrictions.

Consider the focal boundary value problem

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t)+(Tx)(t)=f(t), & t\in[0,1],\\ x^{(i)}(0)=0, & i=0,\dots,k-1,\\ x^{(j)}(1)=0, & j=k,\dots,n-1, \end{cases} \tag{0.1}$$

where $k\in\{1,2,\dots,n-1\}$, $n\geq 2$, $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is a linear boundary operator, $\mathbf{C}[0,1]$ and $\mathbf{L}[0,1]$ are the spaces of continuous and integrable real functions on the interval $[0,1]$ (wish usual norms).

The problems of solving various focal boundary value problems for linear and nonlinear ordinary differential equations and functional differential equations arise in many studies of physical, chemical, and biological processes [1, 2, 8, 13, 15].

For the zero operator T, the boundary value problem

$$\begin{cases} (-1)^{(n-k)}x^{(n)}(t)=f(t), & t\in[0,1],\\ x^{(i)}(0)=0, & i=0,\dots,k-1,\\ x^{(j)}(1)=0, & j=k,\dots,n-1 \end{cases}$$

has a unique solution $x(t)=\int\limits_0^1 G(t,s)f(s)\,ds$, $t\in[0,1]$, where the Green function (see, for example, [8])

$$G(t,s)=\frac{1}{(n-k-1)!}\,\frac{1}{(k-1)!}\int\limits_0^{\min(t,s)}(s-\tau)^{n-k-1}(t-\tau)^{k-1}\,d\tau,\ \ t,s\in[0,1],$$

is non-negative.

1 Integral restrictions

The following simple assertion is a corollary of the Banach fixed-point theorem and the Fredholm property of the boundary value problem.

Assertion 1.1. *If* $\|T\|_{\mathbf{C}\to\mathbf{L}}\leq(n-1)(n-k-1)!(k-1)!$, *then problem* (0.1) *is uniquely solvable.*

Definition 1.1. A linear operator $T:\mathbf{C}[0,1]\to\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

In this work, we weaken the solvability conditions from Assertion 1.1 in the case of positive operator T. For some other boundary value problems similar unimprovable conditions are obtained by R. Hakl, A. Lomtatidze, S. Mukhigulashvili, B. Půža, J. Šremr, and others [6, 9–12].

The norm of a positive operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is defined by the equality

$$\|T\|_{\mathbf{C}\to\mathbf{L}} = \int\limits_0^1 (T\mathbf{1})(t)\,dt,$$

where $\mathbf{1}$ is the unit function.

Theorem 1.1. *Let a non-negative number $\mathcal{T}$ be given. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if the following inequality is valid:*

$$\mathcal{T} \le \min_{0<t<1,\,0<s<1} \frac{G(t,1) + G(1,s) + 2\sqrt{G(t,s)G(1,1)}}{G(t,s)G(1,1) - G(t,1)G(1,s)}\,.$$

Remark 1.1. In (1.1), the expression $G(t,s)G(1,1) - G(t,1)G(1,s)$ is positive for all $t,s \in (0,1)$ because of the kernel $G(t,s)$ is totally positive (see, for example, [7, 14]).

The proof of Theorem 1.1 is based on the following lemma.

Lemma 1.1 ([3])**.** *Let a non-negative number $\mathcal{T}$ be given. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if for all numbers c, d, τ_1, τ_2, $\mathcal{T}_1$, $\mathcal{T}_2$ satisfied the conditions*

$$c,d \in [0,1], \;\; 0 \le \tau_1 \le \tau_2 \le 1,$$
$$\mathcal{T}_1 \ge 0, \;\; \mathcal{T}_2 \ge 0, \;\; \mathcal{T}_1 + \mathcal{T}_2 \le \mathcal{T},$$

the inequality

$$1 + \mathcal{T}_1 G(\tau_1,c) + \mathcal{T}_2 G(\tau_2,d) + \mathcal{T}_1\mathcal{T}_2\big(G(\tau_1,c)G(\tau_2,d) - G(\tau_2,c)G(\tau_1,d)\big) \ge 0$$

is fulfilled.

Theorem 1.2. *Let a non-negative number $\mathcal{T}$ be given and $n = 2k$. Problem* (0.1) *is uniquely solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ with norm $\mathcal{T}$ if and only if the following inequality is valid:*

$$\mathcal{T} \le \frac{2((n/2-1)!)^2}{\max\limits_{0<t<1}\Big(\frac{t^{(n-1)/2}}{n-1} - \int\limits_0^t (t-\tau)^{n/2-1}(1-\tau)^{n/2-1}\,d\tau\Big)} \equiv \mathcal{T}_n.$$

For $n=2$, $n=4$, $n=6$, the numbers $\mathcal{T}_n$ can be calculated exactly. We have

$$\begin{aligned}
\mathcal{T}_2 &= 8,\\
\mathcal{T}_4 &= 66 + 30\sqrt{5} \approx 133.1,\\
\mathcal{T}_6 &= \frac{8}{\frac{t^{5/2}}{5} - \frac{t_6^3(t_6^2 - 5t_6 + 10)}{30}} \approx 2610.5,
\end{aligned}$$

where

$$t_6 = \Big(\frac{C_1 - 1 - \sqrt{27 + 22/C_1 - C_1^2}}{4}\Big)^2,$$
$$C_1 = \sqrt{2C_2 + 9 + 48/C_2}, \quad C_2 = \sqrt[3]{124 + 4\sqrt{97}}\,.$$

For even $n \ge 8$, we obtain sufficient solvability conditions.

Corollary 1.1. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq \frac{(n^2-9)(n^2-1)((n/2-1)!)^2}{3+(n-2)(\frac{n-7}{n-3})^{\frac{n+1}{2}}},$$

then the boundary value problem (0.1) *is uniquely solvable.*

Corollary 1.2. *Let $n = 2k \geq 8$ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be positive. If*

$$\|T\|_{\mathbf{C}\to\mathbf{L}} \leq e^2(n-3)^3((n/2-1)!)^2, \tag{1.1}$$

then the boundary value problem (0.1) *is uniquely solvable.*

Remark 1.2. The sufficient condition in Corollary 1.2 is sharp. The constant e^2 and the exponents cannot be increased in (1.1). Inequality (1.1) significantly improves the solvability condition from Assertion 1.1 (the constant in the solvability conditions is increased approximately $(en)^2$ times for large n).

2 Point-wise restrictions

Consider problem (0.1) for $k = n-1$,

$$\begin{cases} x^{(n)}(t) - (Tx)(t) = f(t), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0, \dots, n-2, \\ x^{(n-1)}(1) = 0. \end{cases} \tag{2.1}$$

Assertion 2.1. *Let $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ be a linear bounded operator. If*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} |(T\mathbf{1})(t)| < (n-2)!n,$$

then problem (2.1) *is uniquely solvable.*

We can improve this assertion for positive operators T.

Lemma 2.1 ([3, Lemma 2.19], [4, Lemma 2], [5, Lemma 1])**.** *Let a non-negative function $p \in L[0,1]$ be given. Problem* (2.1) *is uniquely solvable for all positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ satisfied the equality $T\mathbf{1} = p$ if and only if the focal boundary value problem*

$$\begin{cases} x^{(n)}(t) = p_1(t)x(t_1) + p_2(t)x(t_2), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0, \dots, n-2, \\ x^{(n-1)}(1) = 0 \end{cases}$$

has only the trivial solution for all points $t_1 \leq t_2$, $t_1, t_2 \in [0,1]$ and for all non-negative functions $p_1, p_2 \in L[0,1]$ such that $p_1 + p_2 = p$.

Define

$$k(t) \equiv 1 + P\Big(1 - \frac{t}{n}\Big)\frac{t^{n-1}}{(n-1)!}, \quad t \in [0,1],$$

where P is a constant,

$$G_1(t,s) \equiv \begin{cases} \dfrac{t^{n-1} - (t-s)^{n-1}}{(n-1)!}, & 1 \geq t \geq s \geq 0, \\ \dfrac{t^{n-1}}{(n-1)!}, & 1 \geq s > t \geq 0. \end{cases}$$

Theorem 2.1. *Let a non-negative number P be given. Then the focal boundary value problem* (2.1) *is uniquely solvable for all positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} (T1)(t) \le P$$

if and only if the inequality

$$k(t_2) + P \int\limits_s^1 \big(G_1(t_2,\tau)k(t_1) - G_1(t_1,\tau)k(t_2)\big)\,d\tau > 0$$

is fulfilled for all $0 \le t_1 \le t_2 \le 1$ and all $s \in (0, t_2]$.

We obtain some sufficient solvability conditions for the simplest functional differential equations with one concentrated argument.

Corollary 2.1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative coefficient, $h : [0,1] \to [0,1]$ be a measurable deviated argument.*

Then for $n = 2$, the focal boundary value problem

$$\begin{cases} x^{(n)}(t) = p(t)x(h(t)) + f(t), & t \in [0,1], \\ x^{(i)}(0) = 0, & i = 0,\dots,n-2, \\ x^{(n-1)}(1) = 0 \end{cases} \tag{2.2}$$

is uniquely solvable if

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 16, \quad p(t) \not\equiv 16,$$

where the constant "16" is unimprovable.

For $n = 3$, problem (2.2) *is uniquely solvable if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 58.$$

For $n = 4$, problem (2.2) *is uniquely solvable if*

$$\operatorname*{vrai\,sup}_{t\in[0,1]} p(t) \le 270.$$

Remark 2.1. It seems that for $n = 2$ the best constants "8" and "16" in Theorem 1.2 and Corollary 2.1 are known (see, for example, [3, p. 109] for integral restriction). However, as we know, for higher-order functional differential equations these results are new.

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Asymptotic Properties of $P_\omega(Y_0, Y_1, 0)$-solutions of Second Order Differential Equations with Rapidly and Regularly Varying Nonlinearities

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

Here, $\alpha_0 \in \{-1; 1\}$, functions $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either an interval $[y_i^0, Y_i]$ or an interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$ we will take $y_i^0 > 0$ or $y_i^0 < 0$, respectively.

We also suppose that the function φ_1 is a regularly varying function of index σ_1 as $y \to Y_1$ $(y \in \Delta_{Y_1})$ [4, pp. 10–15], the function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

From the results obtained in the monograph by V. Maric (see, [3, pp. 91–92, p. 117]) it follows the next lemmas.

Lemma 1. *If the function $\varphi : \Delta_Y \to]0, +\infty[$ is differentiable on Δ_Y and the following condition takes place*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{y\varphi'(y)}{\varphi(y)} = l,$$

then $\varphi(y)$ is normalized slowly or regularly varying function as $y \to Y$ in cases $l = 0$, $l \in R \setminus \{0\}$, respectively, and a rapidly varying function as $y \to Y$ in case $l = \pm\infty$.

Lemma 2. *If the function $\varphi : \Delta_Y \to]0, +\infty[$ is measurable, twice continuously differentiable on Δ_Y and satisfies conditions*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \varphi(y) = Z \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty, \quad \lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{\varphi''(y)\varphi(y)}{(\varphi'(y))^2} = 1,$$

then:

1) *the function φ and its first derivative are rapidly varying functions as $y \to Y$;*

2) *there exists a slowly varying function $l_1 : \Delta_Z \to]0, +\infty[$ as the argument tends to Z (Δ_Z is a one-sided neighborhood of Z) such that*

$$\varphi'(y) = \varphi(y) \cdot l_1(\varphi(y));$$

3) *the function* $F(y) = (\varphi(y))^s$ $(s \in R \setminus \{0\})$ *satisfies the condition*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{F''(y)F(y)}{(F'(y))^2} = 1; \tag{3}$$

4) *the function* $F(y) = \int\limits_{y_0^0}^{y} \varphi(\tau)\,d\tau$, *where*

$$y_0^0 = \begin{cases} y_0 & \text{as } \displaystyle\int_{y_0}^{Y} \varphi(\tau)\,d\tau = +\infty, \\ Y & \text{as } \displaystyle\int_{y_0}^{Y} \varphi(\tau)\,d\tau < +\infty, \end{cases}$$

satisfies condition (3).

Lemma 3. *If the function* $\varphi : \Delta_Y \to]0, +\infty[$ *satisfies conditions* (2), *the function* $L : \Delta_Y \to]0, +\infty[$ *is a slowly varying function as* $y \to Y$ $(y \in \Delta_Y)$, *then*

$$\int_{y_0^0}^{y} L(\tau)\varphi(\tau)\,d\tau \sim L(y) \int_{y_0^0}^{y} \varphi(\tau)\,d\tau \quad \text{as } y \to Y,$$

where

$$y_0^0 = \begin{cases} y_0 & \text{as } \displaystyle\int_{y_0}^{Y} L(\tau)\varphi(\tau)\,d\tau = +\infty, \\ Y & \text{as } \displaystyle\int_{y_0}^{Y} L(\tau)\varphi(\tau)\,d\tau < +\infty, \end{cases} \qquad y_0 \in \Delta_Y.$$

Lemma 4. *If* $\varphi_0 : \Delta_{Y_0} \to]0, +\infty[$ *is a rapidly varying function as the argument tends to* Y_0, *the function* $\varphi_1 : \Delta_{Y_1} \to \Delta_{Y_0}$ *satisfies the condition* $\lim\limits_{\substack{y \to Y_1 \\ y \in \Delta_Y}} \varphi_1(y) = Y_0$ *and is a regularly varying function of index* $\sigma \neq 0$ *as the argument tends to* Y_1, *then the function* $\varphi_0(\varphi_1)$ *is also a rapidly varying function as the argument tends to* Y_1.

Lemma 5. *If the rapidly varying as* $y \to Y$ *function* $\varphi : \Delta_Y \to]0, +\infty[$ *is strictly monotone on* Δ_Y *and satisfies the conditions*

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \varphi(y) = Z \in \{0, +\infty\}, \quad \varphi(\Delta_Y) = \Delta_Z,$$

where Δ_Z *is one-sided neighborhood of* Z, *then the function* $\varphi^{-1} : \Delta_Z \to \Delta_Y$ *is a slowly varying function as the argument tends to* Z.

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \leq \lambda_0 \leq +\infty)$ if the following conditions take place

$$y^{(i)} : [t_0, \omega[\longrightarrow \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

In this work we establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in case $\lambda_0=0$ and find asymptotic representations of such solutions and its first order derivatives as $t\uparrow\omega$.

The main result of the work is obtained under the assumption that for $P_\omega(Y_0,Y_1,0)$-solutions of equation (1) there exist the next finite or infinite limit

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}\,.$$

According to the properties of such solutions (see, for example, [1]) we have

$$\begin{aligned}&\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=0,\\ &\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}=-1.\end{aligned}\tag{4}$$

From (4) it follows that function $y'(t)$ is a normalized regularly varying function of index (-1) as $t\uparrow\omega$, that means it can be represented in the form

$$y'(t)=|\pi_\omega(t)|^{-1}L_1(t),$$

where $L_1(t):[t_0,\omega[\,\to\,]-\infty,+\infty[$ is a normalized slowly varying function as $t\uparrow\omega$ [4, pp. 10–15]. It follows that

$$\lim_{t\uparrow\omega}\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(t)|}=Y_1.$$

From the fact that the function L_1 is a normalized slowly varying function, it follows that the function $L_1(t(z))$, where $t(z)$ is the inverted function to the function $z(t)=\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(t)|}$, is also a normalized slowly varying function as $t\uparrow\omega$ because it is a composition of slow and regularly varying functions.

Let us introduce in the following notations.

$$\pi_\omega(t)=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\qquad \theta_1(y)=\varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi(y)=\int\limits_{A_\omega}^{y}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz,\quad A_\omega=\begin{cases}y_0^0 & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=\pm\infty,\\ Y_0 & \text{if } \displaystyle\int\limits_{y_0^0}^{Y_0}|\varphi_0(z)|^{\frac{1}{\sigma_1-1}}\,dz=const,\end{cases}$$

$$\mu_0=\operatorname{sign}(\varphi_0'(y)),\quad Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y).$$

From the indicated conditions onto the function φ_0 we have

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)\in\{0,+\infty\},\quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\Phi''(y)\cdot\Phi(y)}{(\Phi'(y))^2}=1.$$

It follows from this that, like the function φ_0, the function Φ is also a rapidly varying function when the argument tends to Y_0 [4, pp. 10–15]. In addition, the following lemma takes place.

Lemma 6.

1)
$$\Phi(y)=(\sigma_1-1)\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(y)}{\varphi_0'(y)}\,[1+o(1)]\ \ as\ \ y\to Y_0\ \ (y\in\Delta_{Y_0}),$$

from which we have
$$\mu_0\cdot\operatorname{sign}(\Phi(y))=\operatorname{sign}(\sigma_1-1)\ \ as\ \ y\in\Delta_{Y_0}.$$

2) *The function* $\Phi^{-1}(z)\cdot\frac{\Phi'(\Phi^{-1}(z))}{z}$ *is a slowly varying function as* $z\to Z_0$.

Proof. Statement 1) of the lemma follows from the conditions on the function φ_0.

Let us prove statement 2). We have
$$\lim_{z\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_0}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1.$$

So,
$$\lim_{z\to Z_1}\frac{z\cdot(\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z})'}{\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}}=\lim_{y\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}-1=0.$$

The last one means that the function $\Phi_1^{-1}(z)\cdot\frac{\Phi_1'(\Phi_1^{-1}(z))}{z}$ is a slowly varying function as $z\to Z_1$. And the function $\Phi_1^{-1}(z)$ is a slowly varying as $z\to Z_1$ like an inverse function to the rapidly varying one. □

Let's introduce the additional notations.
$$I(t)=\operatorname{sign}(y_1^0)\cdot\int\limits_{B_\omega^2}^{t}\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$
$$B_\omega^2=\begin{cases}\omega & \text{if}\ \ \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\\ b & \text{if}\ \ \displaystyle\int\limits_b^\omega\Big|\pi_\omega(\tau)p(\tau)\theta_1\Big(\frac{\operatorname{sign}(y_1^0)}{|\pi_\omega(\tau)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\end{cases}\qquad b\in[a;\omega[\,.$$

Definition 2. We say that a slowly varying as $z\to Y$ ($z\in\Delta_Y$) function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S as $z\to Y$ if for any continuous differentiable normalized slowly varying as $z\to Y$ ($z\in\Delta_Y$) function $L:\Delta_{Y_i}\to]0;+\infty[$ the next relation is valid
$$\theta(zL(z))=\theta(z)(1+o(1))\ \ \text{as}\ \ z\to Y\ \ (z\in\Delta_Y).$$

Conditions S are satisfied, for example, for such functions as $\ln|y|$, $|\ln|y||^\mu$ ($\mu\in R$), $\ln\ln|y|$.

The following theorem is valid.

Theorem 1. *Let* $\sigma_1\neq1$, *the function* θ_1 *satisfy the condition* S. *For the existence of* $P_\omega(Y_0,Y_1,\lambda_0)$-*solutions of equation* (1), *for which the following finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$ *exists, the* *following conditions are necessary*

$$\alpha_0\pi_\omega(t)y_1^0<0 \ \ as \ \ t\in[a;\omega[\,,$$

$$\lim_{t\uparrow\omega}\frac{y_1^0}{|\pi_\omega(t)|}=Y_1, \tag{5}$$

$$\lim_{t\uparrow\omega}I_2(t)=Z_0, \quad \mu_0(\sigma_1-1)I_2(t)>0 \ \ as \ \ t\in]b;\omega[\,, \tag{6}$$

$$\lim_{t\uparrow\omega}\frac{I_2'(t)\pi_\omega(t)}{\Phi_2'(\Phi_2^{-1}(I_2(t)))\Phi_2^{-1}(I_2(t))}=0. \tag{7}$$

For each such solution the next asymptotic representations take place as $t\uparrow\omega$*:*

$$\Phi_0(y(t))=I_2(t)[1+o(1)], \quad \frac{y'(t)\Phi_0'(y(t))}{\Phi_2(y(t))}=\frac{I_2'(t)}{I_2(t)}[1+o(1)]. \tag{8}$$

Theorem 2. *Let* $\sigma_1\neq1$*, the function* θ_1 *satisfy the condition* S*, the function* $\frac{\pi_\omega(t)\cdot I'(t)}{I(t)}$ *be a normalized slowly varying function as* $t\uparrow\omega$*, the function* $(\frac{\Phi'(y)}{\Phi(y)})$ *be a regularly varying function of some real index as* $y\to Y_0$ $(y\in\Delta_{Y_0})$*. Then in case either*

$$0<\Big|\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_2'(t)}{I_2(t)}\Big|<+\infty, \tag{9}$$

or

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\pm\infty, \quad \mu_0\alpha_0<0, \tag{10}$$

conditions (5)–(7) *are sufficient for the existence of* $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1)*, for which the finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$ *exists.*

During the proof of Theorem 2, equation (1) is reduced by a special transformation to the equivalent system of quasilinear differential equations. The limit matrix of coefficients of this system has real eigenvalues of different signs. We obtain that for this system of differential equations all the conditions of Theorem 2.2 in [2] take place. According to this theorem, the system has a one-parameter family of solutions $\{z_i\}_{i=1}^2:[x_1,+\infty[\,\to\mathbb{R}^2$ $(x_1\geq x_0)$, that tends to zero as $x\to+\infty$.

Any solution of the family gives raise to such a solution y of equation (1) that, together with its first derivative, admits the asymptotic images (8) as $t\uparrow\omega$. From these images and conditions (5)–(7), (9), (10) it follows that these solutions are $P_\omega(Y_0,Y_1,0)$-solutions.

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Boundary Value Problems for Systems of Difference-Algebraic Equations

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We investigate the problem of finding bounded solutions [2, 3, 5]

$$z(k) \in \mathbb{R}^n, \quad k \in \Omega := \{0, 1, 2, \dots, \omega\}$$

of linear Noetherian ($n \neq \upsilon$) boundary value problem for a system of linear difference-algebraic equations [2, 5]

$$A(k)z(k+1) = B(k)z(k) + f(k), \quad \ell z(\,\cdot\,) = \alpha, \quad \alpha \in \mathbb{R}^{\upsilon}; \tag{1}$$

here $A(k),\ B(k) \in \mathbb{R}^{m \times n}$ are bounded matrices and $f(k)$ are real bounded column vectors,

$$\ell z(\,\cdot\,) : \mathbb{R}^n \to \mathbb{R}^{\upsilon}$$

is a linear bounded vector functional defined on a space of bounded functions. We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m = n$. It can be square but singular. The problem of finding bounded solutions $z(k)$ of a boundary value problem for a linear non-degenerate

$$\det B(k) \neq 0, \quad k \in \Omega$$

system of first-order difference equations

$$z(k+1) = B(k)z(k) + f(k), \quad \ell z(\,\cdot\,) = \alpha \in \mathbb{R}^{\upsilon}$$

was solved by A. A. Boichuk [2]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. A. Boichuk. We investigate the problem of finding bounded solutions to linear Noetherian boundary value problem for a system of linear difference-algebraic equations (1) in case

$$1 \leq \text{rank}\ A(k) = \sigma_0, \quad k \in \Omega.$$

As it is known [1, 10], any $(m \times n)$-matrix $A(k)$ can be represented in a definite basis in the form

$$A(k) = R_0(k) \cdot J_{\sigma_0} \cdot S_0(k), \quad J_{\sigma_0} := \begin{pmatrix} I_{\sigma_0} & O \\ O & O \end{pmatrix};$$

here, $R_0(k)$ and $S_0(k)$ are nonsingular matrices. The nonsingular change of the variable

$$y(k+1) = S_0(k)z(k+1)$$

reduces system (1) to the form [11]

$$A_1(k)\varphi(k+1) = B_1(k)\varphi(k) + f_1(k); \tag{2}$$

Under the condition [10], when matrices $A_1^+(k)B_1(k)$ and column vectors $A_1^+(k)f_1(k)$, are bounded and also
$$P_{A^*}(k) \neq 0, \;\; P_{A_1^*}(k) \equiv 0, \tag{3}$$
we arrive at the problem of construction of solutions of the linear difference-algebraic system
$$\varphi(k+1) = A_1^+(k)B_1(k)\varphi(k) + \mathfrak{F}_1(k, \nu_1(k)), \;\; \nu_1(k) \in \mathbb{R}^{\rho_1}; \tag{4}$$
here,
$$\mathfrak{F}_1(k, \nu_1(k)) := A_1^+(k)f_1(k) + P_{A_{\varrho_1}}(k)\nu_1(k),$$
$\nu_1(k) \in \mathbb{R}^{\rho_1}$ is an arbitrary bounded vector function, $A_1^+(k)$ is a pseudoinverse (by Moore–Penrose) matrix [3]. In addition, $P_{A_1^*(k)}$ is a matrix-orthoprojector [3]:
$$P_{A_1^*}(k) : \mathbb{R}^{\sigma_0} \to \mathbb{N}(A_1^*(k)),$$
$P_{A_{\rho_1}}(k)$ is an $(\rho_0 \times \rho_1)$-matrix composed of ρ_1 linearly independent columns of the $(\rho_0 \times \rho_0)$-matrix-orthoprojector:
$$P_{A_1}(k) : \mathbb{R}^{\rho_0} \to \mathbb{N}(A_1(k)).$$
By analogy with the classification of pulse boundary-value problems [3, 6, 7] we say in the (3), provided that the matrices $A_1^+(k)B_1(k)$ and column vectors $A_1^+(k)f_1(k)$ are bounded, that, for the linear difference-algebraic system (1), the first-order degeneration holds. Thus, the following lemma is proved [11].

Lemma 1. *For the first-order degeneration difference-algebraic system* (1) *having a solution of the form*
$$z(k, c_{\rho_0}) = X_1(k)\, c_{\rho_0} + K[f(j), \nu_1(j)](k), \;\; c_{\rho_0} \in \mathbb{R}^{\rho_0};$$
which depends on an arbitrary continuous vector-function $\nu_1(k) \in \mathbb{R}^{\rho_1}$, where $X_1(k)$ is a fundamental matrix, $K[f(j), \nu_1(j)](k)$ is the generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1).

Denote the vector
$$\nu_1(k) := \Psi_1(k)\gamma, \;\; \gamma \in \mathbb{R}^{\theta};$$
here, $\Psi_1(k) \in \mathbb{R}^{\rho_1 \times \theta}$ is an arbitrary bounded full rank matrix. Generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1) of the form
$$K\big[f(j), \nu_1(j)\big](k) = K\big[f(j)\big](k) + K\big[\Psi_1(j)\big](k)\,\gamma;$$
here,
$$K\big[\Psi_1(j)\big](k) := S_0^{-1}(k-1)P_{D_{\rho_0}}\mathcal{K}\big[\Psi_1(s))\big](k),$$
and
$$\begin{aligned}\mathcal{K}\big[\Psi_1(j)\big](0) := 0, \quad \mathcal{K}\big[\Psi_1(j)\big](1) := P_{A_{\rho_1}}(0)\Psi_1(0),&\\ \mathcal{K}\big[\Psi_1(j)\big](2) := A_1^+(1)B_1(1)\mathcal{K}\big[\Psi_1(j)\big](1) + P_{A_{\rho_1}}(1)\Psi_1(1), \; \dots \;,&\\ \mathcal{K}\big[\Psi_1(j)\big](k+1) := A_1^+(k)B_1(k)\mathcal{K}\big[\Psi_1(j)\big](k) + P_{A_{\rho_1}}(k)\Psi_1(k).&\end{aligned}$$
Denote the matrix
$$\mathcal{D}_1 := \big\{Q_1; \ell K\big[\Psi_1(j)\big](\,\cdot\,)\big\} \in \mathbb{R}^{\upsilon \times (\rho_0+\theta)}.$$

Substituting the general solution of the system of linear difference-algebraic equations (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$\mathcal{D}_1\,\check{c}=\alpha-\ell K\big[A^+(j)f(j)\big](\,\cdot\,),\quad \check{c}:=\operatorname{col}(c_{\rho_0},\gamma)\in\mathbb{R}^{\rho_0+\theta}. \tag{5}$$

Equation (5) is solvable iff

$$P_{\mathcal{D}_1^*}\big\{\alpha-\ell K\big[f(j)\big](\,\cdot\,)\big\}=0. \tag{6}$$

Here, $P_{\mathcal{D}_1^*}$ is a matrix-orthoprojector:

$$P_{\mathcal{D}_1^*}:\ \mathbb{R}^{\upsilon}\to\mathbb{N}(\mathcal{D}_1^*).$$

In this case, the general solution of equation (5)

$$\check{c}={\mathcal{D}_1}^+\big\{\alpha-\ell K\big[f(j)\big](\,\cdot\,)\big\}+P_{\mathcal{D}_1}\,\delta,\ \ \delta\in\mathbb{R}^{\rho_0+\theta}$$

determines the general solution of the boundary-value problem (1)

$$\begin{aligned}z(k,\delta)=\big\{X_1(k);K\big[\Psi_1(j)\big](k)\big\}\mathcal{D}_1^+\big\{\alpha-\ell K\big[f(j)\big](\,\cdot\,)\big\}&\\ +K\big[f(j)\big](k)+\big\{X_1(k);K\big[\Psi_1(j)\big](k)\big\}P_{\mathcal{D}_1}\,\delta.&\end{aligned}$$

Here, $P_{\mathcal{D}_1}$ is a matrix-orthoprojector:

$$P_{\mathcal{D}_1}:\mathbb{R}^{\rho_0+\theta}\to\mathbb{N}(\mathcal{D}_1).$$

Thus the following theorem is valid.

Theorem 1. *The problem of finding bounded solutions of a system of linear difference-algebraic equations* (1) *in the case of first-order degeneracy, under condition* (3)*, in the case of first-order degeneracy for a fixed full rank bounded matrix* $\Psi_1(k)$*, has a solution of the form*

$$z(k,c_{\rho_0})=X_1(k)\,c_{\rho_0}+K[f(j),\nu_1(j)](k),\ \ c_{\rho_0}\in\mathbb{R}^{\rho_0}.$$

Under condition (6) *and only under it, the general solution of the difference-algebraic boundary value problem* (1)

$$z(k,c_r)=X_r(k)c_r+G\big[f(j);\Psi_1(j);\alpha\big](k),\ \ c_r\in\mathbb{R}^r$$

is determined by the Green operator of a difference-algebraic boundary value problem (1)

$$G\big[f(j);\Psi_1(j);\alpha\big](k):=K\big[f(j)\big](k)+\big\{X_1(k);K\big[\Psi_1(j)\big](k)\big\}\mathcal{D}_1^+\big\{\alpha-\ell K\big[f(j)\big](\,\cdot\,)\big\}.$$

The matrix $X_r(k)$ *is composed of* r *linearly independent columns of the matrix*

$$\big\{X_1(k);K\big[\Psi_1(j)\big](k)\big\}P_{\mathcal{D}_1}.$$

Under condition $P_{\mathcal{D}_1^*}\neq 0$, we say that the difference-algebraic boundary-value problem (1) in the case of first-order degeneracy is a critical case, and vice versa: under condition $P_{Q_1^*}\neq 0$, $P_{\mathcal{D}_1^*}=0$, we say that the difference-algebraic boundary-value problem (1) is reduced to the non-critical case.

The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [2–4, 9] onto nonlinear difference-algebraic boundary-value problems. On the other hand, in the case of nonsolvability, the difference-algebraic boundary-value problems can be regularized analogously [8, 12].

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Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems with the Trivial Averaging of Coefficient Matrix

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Until the middle of the 20th century, the study of periodic solutions of periodic differential systems was based on the hypothesis of the commensurability of the periods of a solution and a system. At the same time, N. D. Papaleksi carried out work on the study of parametric effects on dual-circuit electrical systems. He demonstrated the possibility of excitation of oscillations at a frequency incommensurable with the frequency of changes in the system parameters [8]. In 1950, H. Massera showed that periodic differential systems can have periodic solutions such that the period of a solution is incommensurable with the period of the system. His work [7] laid the foundation for a new direction in the qualitative theory of differential equations which was further developed in the studies of J. Kurzweil and O. Vejvoda [5], N. P. Erugin [2], I. V. Gaĭshun [3], E. I. Grudo [4] and others. Subsequently, such periodic solutions were called strongly irregular [1, p. 16], and the oscillations described by them were called asynchronous. The problem of constructing of asynchronous modes can be formulated as the problem of controlling of the spectrum of irregular oscillations.

First we present the necessary definitions from the theory of almost periodic (on Bohr) functions [6]. Let f be a real continuous function. The function f is called almost periodic if, for an arbitrary positive ε, the set of its ε-almost-periods is relatively dense. Each almost periodic function f has an average value

$$\widehat{f} = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T f(s)\,ds.$$

Put $\widetilde{f}(t) = f(t) - \widehat{f}$. The function $\widetilde{f}$ will be called the oscillating part of an almost periodic function f. Note that in contrast to periodic functions, there exist almost periodic functions $\widetilde{f}$ whose integral is not a almost periodic. A real number λ such that

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \exp(-i\lambda s) f(s)\,ds \neq 0$$

is called the Fourier exponent (or frequency) of an almost periodic function f. The set of all frequencies forms the set of Fourier exponents (frequency spectrum) of the function f. The module (frequency module) $\mathrm{Mod}(f)$ of an almost periodic function f is the smallest additive group of real numbers containing all the Fourier exponents of this function.

Let $g(t,x)$ be a vector function that is almost periodic in t uniformly with respect to x from some compact set. An almost periodic solution $x(t)$ of the system of ordinary differential equations

$$\dot{x} = g(t,x)$$

will be called strongly irregular if the intersection of the frequency modules of the solution and the right-hand side of the system is trivial, i.e.

$$\operatorname{Mod}(x) \cap \operatorname{Mod}(g) = \{0\}.$$

Let $P(t)$ be a continuous matrix. Denote by $\operatorname{rank}_{col} P$ the column rank of the matrix $P(t)$, i.e. $\operatorname{rank}_{col} P$ is the largest number of its linearly independent columns.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $A(t)$ is a continuous almost periodic $n \times n$-matrix, B is a constant $n \times n$-matrix. We assume that the linear state feedback control

$$u = U(t)x \tag{2}$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ is used, $\operatorname{Mod}(U) \subseteq \operatorname{Mod}(A)$.

The problem of finding a matrix $U(t)$ (the feedback factor) such that the closed-loop system

$$\dot{x} = (A(t) + BU(t))x$$

has a strongly irregular almost periodic solutions with a given frequency spectrum L (the objective set) is called the control problem for the spectrum of irregular oscillations with objective set L (control problem of asynchronous spectrum).

Note first that in the case of a non-singular matrix B, the solution of this problem is not difficult. Therefore, we will assume that the matrix B is a singular,

$$\operatorname{rank} B = r < n \;\; (n - r = d).$$

By $B_{d,n}$ and $B_{r,n}$ we denote the matrices consisting of the first d rows and the remaining r rows of the matrix B, respectively. One can assume that the first d rows of the matrix B are zero, i.e.,

$$\operatorname{rank} B_{d,n} = 0, \tag{3}$$

because otherwise such a form can be achieved by a linear nonsingular stationary transformation. Note that the rank of the matrix $B_{r,n}$ is equal to r as well.

We will also assume that the matrix $A(t)$ has a zero mean value, i.e.,

$$\widehat{A} = \lim_{T \to \infty} \frac{1}{T} \int\limits_0^T A(s)\, ds = 0, \tag{4}$$

We give conditions for the solvability of the control problem of asynchronous spectrum for system (1).

Let

$$L = \{\lambda_1, \lambda_2, \dots\}$$

be the objective frequency set.

Taking into account the structure of the matrix B, we represent the coefficient matrix $A(t)$ in a block form. Let $A_{d,d}(t)$ and $A_{r,d}(t)$ be its left upper and lower blocks, and let $A_{d,r}(t)$ and $A_{r,r}(t)$ be the right upper and lower blocks (the subscripts show the block dimension).

The following theorem holds.

Theorem. *Let the first d rows of the matrix B in system* (1) *be zero and the remaining rows be linearly independent, let the coefficient matrix $A(t)$ have a zero mean value, and let the following estimates hold:*

(i) $\operatorname{rank}_{col} A_{d,r} = r_1 < r$;

(ii) $|L| \le [(r - r_1)/2]$.

Then the control problem for the spectrum of irregular oscillations with objective set L for system (1) *with feedback* (2) *is solvable.*

Remark. *Estimates* (i) *and* (ii) *in the theorem are necessary and sufficient conditions for the solvability of the investigated problem for the class of systems* (1) *under assumptions* (3), (4).

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On Periodic Boundary Value Problem for a Certain Planar System of Nonlinear Ordinary Differential Equations

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On an interval $[0,\omega]$ we consider the system

$$u_1' = p_1(t)|u_2|^{\lambda_1}\operatorname{sgn} u_2 + q_1(t), \quad u_2' = p_2(t)|u_1|^{\lambda_2}\operatorname{sgn} u_1 + q_2(t) \tag{1}$$

subject to the boundary conditions

$$u_1(0) = u_1(\omega) + c_1, \quad u_2(0) = u_2(\omega) + c_2. \tag{2}$$

Here we suppose that $p_i, q_i \in L([0,\omega])$, $c_i \in \mathbb{R}$, $i = 1,2$ and

$$\lambda_1 > 0, \quad \lambda_1\lambda_2 = 1. \tag{3}$$

In the linear case, i.e., where $\lambda_1 = 1$ (and $\lambda_2 = 1$), problem (1), (2) as well as its particular case, scalar problem, are studied in sufficient detail. As for the general case, as far as we know, there is still a broad field for further investigations. The aim of the present paper is to fill the existing gap in a certain sense.

Along with (1), (2), we consider also the corresponding "homogeneous" problem

$$u_1' = p_1(t)|u_2|^{\lambda_1}\operatorname{sgn} u_2, \quad u_2' = p_2(t)|u_1|^{\lambda_2}\operatorname{sgn} u_1, \tag{1_0}$$

$$u_1(0) = u_1(\omega), \quad u_2(0) = u_2(\omega). \tag{2_0}$$

It has been proved recently in [1] that if (3) holds and (1_0), (2_0) has no non-trivial solution, then for any $q_1, q_2 \in L([0,\omega])$ and $c_1, c_2 \in \mathbb{R}$, problem (1), (2) possesses at least one solution. In other words, the Fredholm property, which is well-known for the linear case, remains true (except uniqueness).

Introduce the definition.

Definition. Let (3) hold and $p_1, p_2 \in L([0,\omega])$. We say that the vector function (p_1, p_2) belongs to the set $V^-(\omega,\lambda_1)$ if for any $(u_1,u_2) \in AC([0,\omega];\mathbb{R}^2)$ such that

$$u_1'(t) = p_1(t)|u_2(t)|^{\lambda_1}\operatorname{sgn} u_2(t), \quad u_2'(t) \geq p_2(t)|u_1(t)|^{\lambda_2}\operatorname{sgn} u_1(t),$$

for a.e. $t \in [0,\omega]$, and
$$u_1(0) = u_1(\omega), \quad u_2(0) \geq u_2(\omega),$$
the inequality
$$u_1(t) \leq 0 \ \text{ for } t \in [0,\omega]$$
is fulfilled.

Remark 1. It is not difficult to verify that if $p_1 \not\equiv 0$ on $[0,\omega]$ and $(p_1,p_2) \in V^-(\omega,\lambda_1)$, then problem (1_0), (2_0) has no non-trivial solutions. Consequently, (1), (2) is solvable, however in spite of linear problem it is not known whether or not the solution of (1), (2) is unique.

Below we suppose also that
$$p_1(t) \geq 0 \ \text{ for a.e. } t \in [0,\omega] \text{ and } p_1 \not\equiv 0 \text{ on } [0,\omega]. \tag{4}$$

The next theorem states that in some cases problem (1), (2) has no more than one solution.

Theorem 1. *Let* (3) *and* (4) *hold,* $(p_1,p_2) \in V^-(\omega,\lambda_1)$, $c \geq 0$, $q \in L([0,\omega])$ *and* $q(t) \geq 0$ *for a.e.* $t \in [0,\omega]$. *Let, moreover,*
$$c + \operatorname{mes}\big\{t \in [0,\omega] : \ q(t) > 0\big\} > 0.$$
Then the problem
$$\begin{gathered}u_1' = p_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \quad u_2' = p_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 - q(t),\\ u_1(0) = u_1(\omega), \quad u_2(0) = u_2(\omega) - c\end{gathered}$$
is uniquely solvable and its solution (u_1,u_2) *satisfies*
$$u_1(t) > 0 \ \text{ for } t \in [0,\omega].$$

Next, let us present necessary and sufficient conditions for the inclusion $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Theorem 2. *Let* (3) *and* (4) *be fulfilled. Then the inclusion* $(p_1,p_2) \in V^-(\omega,\lambda_1)$ *holds if and only if there exists* $(\gamma_1,\gamma_2) \in AC([0,\omega];\mathbb{R}^2)$ *satisfying*
$$\begin{gathered}\gamma_1(t) > 0 \ \text{ for } t \in [0,\omega],\\ \gamma_1'(t) = p_1(t)|\gamma_2(t)|^{\lambda_1} \operatorname{sgn} \gamma_2(t), \quad \gamma_2'(t) \leq p_2(t)\gamma_1^{\lambda_2}(t) \ \text{ for a.e. } t \in [0,\omega],\\ \gamma_1(0) \geq \gamma_1(\omega), \quad \frac{\gamma_2(\omega)}{\gamma_1^{\lambda_2}(\omega)} \geq \frac{\gamma_2(0)}{\gamma_1^{\lambda_2}(0)},\end{gathered}$$
and
$$\gamma_1(0) - \gamma_1(\omega) + \frac{\gamma_2(\omega)}{\gamma_1^{\lambda_2}(\omega)} - \frac{\gamma_2(0)}{\gamma_1^{\lambda_2}(0)} + \operatorname{mes}\big\{t \in [0,\omega] : \ \gamma_2'(t) < p_2(t)\gamma_1^{\lambda_2}(t)\big\} > 0.$$

The following corollary follows from Theorem 2 with $(\gamma_1,\gamma_2) \overset{\text{def}}{=} (1,0)$.

Corollary 1. *Let* (3) *and* (4) *hold,* $p_2(t) \geq 0$ *for* $t \in [0,\omega]$, *and* $p_2 \not\equiv 0$ *on* $[0,\omega]$. *Then* $(p_1,p_2) \in V^-(\omega,\lambda_1)$.

Corollary 2. *Let* (3) *and* (4) *hold and let there exist* $\varphi \in AC([0,\omega])$ *such that*

$$\int_0^\omega p_1(s)|\varphi(s)|^{\lambda_1} \operatorname{sgn} \varphi(s)\, ds \le 0, \tag{5}$$

$$\varphi(0) \le \varphi(\omega), \tag{6}$$

and

$$\Phi(t) \stackrel{\text{def}}{=} \varphi'(t) + \lambda_2 p_1(t)|\varphi_1(t)|^{\lambda_1+1} - p_2(t) \le 0 \ \textit{ for a.e. } t \in [0,\omega].$$

Let, moreover, either one of inequalities (5) *or* (6) *hold in a strong sense or* $\operatorname{mes}\{t \in [0,\omega] :\ \Phi(t) < 0\} > 0$. *Then* $(p_1, p_2) \in V^-(\omega, \lambda_1)$.

Theorem 1 with suitable choice of vector function (γ_1, γ_2) implies the following efficient conditions for inclusion $(p_1, p_2) \in V^-(\omega, \lambda_1)$.

Theorem 3. *Let* (3) *and* (4) *hold,* $p_2 \not\equiv 0$ *on* $[0,\omega]$,

$$\|p_1\|_L \|[p_2]_-\|_L^{\lambda_1} < 2^{\lambda_1+1}, \tag{7}$$

and

$$\|[p_2]_+\|_L > \|[p_2]_-\|_L \Big(1 - \frac{1}{2^{\lambda_1+1}}\, \|p_1\|_L\, \|[p_2]_-\|_L^{\lambda_1}\Big)^{-\lambda_2}. \tag{8}$$

Then the inclusion $(p_1, p_2) \in V^-(\omega, \lambda_1)$ *holds.*

Remark 2. Assumption (8) in Theorem 3 is optimal and cannot be weakened to the assumption

$$\|[p_2]_+\|_L \ge \|[p_2]_-\|_L \Big(1 - \frac{1}{2^{\lambda_1+1}}\, \|p_1\|_L\, \|[p_2]_-\|_L^{\lambda_1}\Big)^{-\lambda_2}. \tag{9}$$

Nevertheless, it is possible to prove the following theorem.

Theorem 4. *Let* (3)*,* (4)*,* (7)*, and* (9) *hold. Let, moreover, either*

$$p_1(t) > 0 \ \textit{ for a.e. } t \in [0,\omega]$$

or

$$\lambda_1 < 1 \ \textit{ and } \ p_1^{\frac{2}{\lambda_1+1}} \notin L([0,\omega]).$$

Then $(p_1, p_2) \in V^-(\omega, \lambda_1)$.

Theorem 5. *Let* (3) *and* (4) *hold,*

$$c \stackrel{\text{def}}{=} \frac{1}{\|p_1\|_L} \int_0^\omega p_2(s)\, ds > 0, \tag{10}$$

and

$$\int_0^\omega [p_2(s) - cp_1(s)]_+\, ds \le \Big(\frac{c}{\lambda_2}\Big)^{\frac{1}{\lambda_1+1}}. \tag{11}$$

Then $(p_1, p_2) \in V^-(\omega, \lambda_1)$.

Example. Let $\omega = 2\pi$, $p_1 \equiv 1$ and $p_2(t) \stackrel{\text{def}}{=} a - b\cos t$ for $t \in [0,\omega]$, where $a > 0$. Then it is clear that

$$\int\limits_0^{\omega} p_2(s)\,ds = a\omega \ \text{ and } \ c = a$$

with c defined by (10). Assumption (11) has the form $|b| \leq \frac{1}{2}\,(a\lambda_2^{-1})^{\frac{1}{\lambda_1+1}}$. On the other hand, if $a \geq |b|$, then the conditions of Corollary 1 are obviously satisfied. Finally, if (3) holds and

$$|b| \leq \max\Big\{a, \frac{1}{2}\,(a\lambda_2^{-1})^{\frac{1}{\lambda_1+1}}\Big\},$$

then the vector function (p_1,p_2) defined above belongs to the set $V^-(\omega,\lambda_1)$.

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Decaying Solutions of Delay Differential Equations

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1 Introduction

Consider the differential equation with damping term

$$x'' = h(t, x(t), x(\gamma(t)))x'(t) + f(t, x(\tau(t)), x(t)), \tag{1.1}$$

where:

1. the functions γ, τ are continuous functions on $[t_0, \infty)$ such that $\gamma(t) \geq t_0, \tau(t) \geq t_0$ and $\lim\limits_{t\to\infty} \gamma(t) = \lim\limits_{t\to\infty} \tau(t) = \infty$;
2. the function h is a continuous function on $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$;
3. the function f is a continuous function on $[t_0, \infty) \times \mathbb{R} \times \mathbb{R}$ and

   $$0 < f(t, u, v) \leq b(t) \ \text{ for any } \ (u, v) \in (0, 1] \times (0, 1], \tag{Hp1}$$

   where b is a positive continuous function on $[t_0, \infty)$.

Let x be a solution of (1.1) and denote by H_x the function

$$H_x(t) = \exp\bigg(-\int\limits_{t_0}^{t} h\big(r, x(r), x(\gamma(r))\big)\,dr \bigg).$$

Hence (1.1) is equivalent to the functional equation

$$(H_x(t)x'(t)) = H_x(t) f\big(t, x(\tau(t)), x(t)\big), \tag{1.2}$$

that is an equation in which the differential operator depends also on the state x. Equations of this type model reaction-diffusion problems with non-constant diffusivity, see, e.g., the papers [6, 14] and the references therein.

A prototype of (1.2) is the nonlinear equation

$$(p(t)g(x)x'(t))' = f(t, x(\tau(t)), \tag{1.3}$$

where p is a positive continuously differentiable function on $[t_0, \infty)$ and g is a continuously positive differentiable function on $\mathbb{R}$. Equation (1.3) includes the well-known Thomas–Fermi equation, as well as the Schroedinger–Persico equation, which occur in the study of atomic fields, see [17]. Moreover, (1.3) arises also in some mechanical problems as the law of angular momentum conservation when the field strength is time dependent, see [11].

Our aim here is to present some results concerning solutions x of (1.1) satisfying

$$x(t) > 0, \ \ x'(t) < 0 \ \text{ for large } t. \tag{1.4}$$

Further, the asymptotic behavior is also examined, jointly with some comments and open problems. These results are taken from [7] and are here presented without proofs.

Solutions of (1.1) satisfying (1.4) are usually called *Kneser solutions*. The Kneser existence problem and the asymptotic decay of Kneser solutions have been deeply studied in the case without deviating arguments. We refer the reader to the monograph [9], the papers [1,3] and the references therein. In the general case of equations with deviating arguments, we refer to the books [8,12], the papers [10,16] and the references therein.

2 Main results

Since $\lim\limits_{t\to\infty} \gamma(t) = \lim\limits_{t\to\infty} \tau(t) = \infty$, there exists $\widetilde{t} \geq t_0$ such that $\gamma(t) \geq t_0$ and $\tau(t) \geq t_0$ for any $t \in [\widetilde{t}, \infty)$. Thus, define t_1 such that

$$t_1 = \inf \big\{ \widetilde{t} \geq t_0 : \ \min\{\gamma(t), \tau(t)\} \geq t_0 \text{ on } [\widetilde{t}, \infty) \big\}. \tag{2.1}$$

Our main result is the following.

Theorem 2.1. *Assume that there exist two functions $\lambda \in C^1(I, \mathbb{R}^+)$, $\theta \in C(I, \mathbb{R})$ such that:*

(i_1) *for any $t \geq t_0$ and $0 \leq u \leq 1$, $0 \leq v \leq 1$*

$$h(t, u, v) \geq \frac{\lambda'(t)}{\lambda(t)}\,. \tag{2.2}$$

(i_2)

$$\int\limits_{t_1}^{\infty} \lambda(s) \int\limits_{s}^{\infty} \theta(r) b(r) \, \mathrm{d}r \, \mathrm{d}s < \infty. \tag{2.3}$$

(i_3) *for every $t > t_0$*

$$\lambda(t)\theta(t) > t_0. \tag{2.4}$$

Then the BVP (1.1), (1.4) *has at least one solution.*

Theorem 2.1 requires the existence of two auxiliary functions, namely λ and θ, satisfying certain properties. The following results give examples of such functions.

Corollary 2.1. *Assume that for some* $n \in \mathbb{N} \cup \{0\}$,

$$\int_{t_0}^{\infty} s^{-n} \int_{s}^{\infty} r^n b(r)\,\mathrm{d}r\,\mathrm{d}s < \infty.$$

If for any $t \ge t_0$ *and* $0 \le u \le 1$, $0 \le v \le 1$,

$$h(t,u,v) \ge -\frac{n}{t}\,, \tag{2.5}$$

then the BVP (1.1), (1.4) *has at least one solution.*

Proof. The assertion follows from Theorem 2.1 by choosing

$$\lambda(t) = t^{-n}, \quad \theta(t) = (t_0+1)t^n. \qquad \square$$

Corollary 2.2. *Assume that for some* $M > 0$,

$$\int_{t_0}^{\infty} \mathrm{e}^{-Ms} \int_{s}^{\infty} \mathrm{e}^{Mr} b(r)\,\mathrm{d}r\,\mathrm{d}s < \infty.$$

If for any $t \ge t_0$ *and* $0 \le u \le 1$, $0 \le v \le 1$,

$$h(t,u,v) \ge -M, \tag{2.6}$$

then BVP (1.1), (1.4) *has at least one solution.*

Proof. The assertion follows from Theorem 2.1 by choosing

$$\lambda(t) = \mathrm{e}^{-Mt}, \quad \theta(t) = (t_0+1)\mathrm{e}^{Mt}. \qquad \square$$

Remark 2.1. Observe that the assumption (2.5) in Corollary 2.1 and the assumption (2.6) in Corollary 2.2 permit us to choose damping terms which take negative values.

Remark 2.2. Theorem 2.1 does not require superlinear conditions (or sublinear conditions) on the forcing term f. Hence, it may be applicable in a wide variety of cases.

Remark 2.3. The proof of Theorem 2.1 is based on a fixed point theorem for multivalued operators which arises from [4]. The main advantage of this approach is that the explicit form of the fixed point operator is not needed, because the topological properties, like the compactness and continuity of the fixed point operator are obtained directly from the *a-priori* bounds for solutions of a suitable associated BVP.

In the sequel, consider the special case of (1.1)

$$x''(t) = h(t, x(t), x(\gamma(t)))x'(t) + \psi(t, x(\tau(t))), \tag{2.7}$$

where the functions γ, τ and h are as in (1.1), τ is a delay and ψ is a continuous function on $[t_0,\infty) \times \mathbb{R}$ such that

$$0 < \psi(t,u) \le b(t) \ \text{ for any } \ u \in (0,1]. \tag{2.8}$$

Observe that in (2.7) the forcing term ψ depends on state x at time $\tau(t)$, but does not depend on x at time t.

Theorem 2.2. *Assume that:*

(i_1)
$$\int\limits_{t_0}^{\infty} t\,b(t)\,\mathrm{d}t < \infty. \tag{2.9}$$

(i_2) $\tau(t) < t$.

(i_3) *The function* h *is nonnegative on* $[t_0,\infty)\times[0,1]\times[0,1]$.

Then the equation (2.7) *has Kneser solutions* x *which satisfy*

$$x(t)x'(t) < 0 \;\; \text{on } t \in [t_1,\infty), \tag{2.10}$$

where t_1 *is given in* (2.1) *and*

$$\lim_{t\to\infty} x(t) = 0. \tag{2.11}$$

Theorem 2.2 shows a discrepancy between equations with or without delay, which is illustrated by the following example.

Example 2.1. Consider the equation

$$x''(t) = g(t)\sqrt{x^2(t) + x^2(\gamma(t))}\;x'(t) + \mathrm{e}^{-t}x(t-\pi), \tag{2.12}$$

where g is a nonnegative continuous function on $[t_0,\infty)$. In view of Theorem 2.2, equation (2.12) has Kneser solutions which satisfy (2.10) and (2.11). Observe that when $g \equiv 0$ on $[t_0,\infty)$, then any Kneser solution of the corresponding linear equation without delay

$$x''(t) = e^{-t}x(t)$$

does not converge to zero as $t \to \infty$, see, e.g. [15, Section 4].

3 Open problems

Open problem 1. Consider the Emden–Fowler equation

$$x''(t) = b(t)|x(t)|^{\beta}\operatorname{sgn} x(t) \tag{3.1}$$

and the corresponding equation with deviating argument

$$x''(t) = b(t)|x(\tau(t))|^{\beta}\operatorname{sgn} x(\tau(t)), \tag{3.2}$$

where b is a positive function on $[t_0,\infty)$ and $\tau(t) < t$.

First observe that if $\beta > 1$ and b is positive, then equation (3.1) always has Kneser solutions. Moreover, if in addition

$$\int\limits_{t_0}^{\infty} sb(s)\,\mathrm{d}s < \infty,$$

then (3.1) does not have Kneser solutions which tend to zero as $t \to \infty$, see, e.g., [3]. If $\tau(t) < t$, then this result may fail for (3.2) as Theorem 2.2 illustrates.

In the sublinear case, that is $0 < \beta < 1$, there might exist equations of type (3.1) without Kneser solutions. For instance, if

$$\liminf_{t\to\infty} t^2 b(t) > 0, \tag{3.3}$$

then (3.1) does not have Kneser solution, see [9, Corollary 17.3]. On the other hand, from Corollary 2.1 with $n = 0$ we get that the equation

$$x''(t) = \frac{1}{t^2 \log t}\, |x(\tau(t))|^\beta \operatorname{sgn} x(\tau(t)), \quad t \geq 2, \tag{3.4}$$

has Kneser solutions. For (3.4) we have

$$\liminf_{t\to\infty} t^2 b(t) = \liminf_{t\to\infty} \frac{1}{\log t} = 0.$$

Thus, it is an open problem if the Kiguradze condition (3.3) is sufficient for the nonexistence of Kneser solutions of (3.2) when $0 < \beta < 1$ and $\tau(t) - t \not\equiv 0$. Finally, observe that if

$$\tau(t) < t \ \text{ and } \ 0 < \beta < 1,$$

then, in view of Theorem 2.2, equation (3.4) has Kneser solutions which tend to zero as $t \to \infty$.

Open problem 2. Kneser solutions which are decaying to zero as $t \to \infty$ may have a different asymptotic behavior, as the following example illustrates.

Example 3.1. Equation

$$\begin{aligned} x''(t) = {} & \frac{(t^3 - 2\mathrm{e}^t)(t^2 + t\ln x - 2\mathrm{e}^{\frac{t}{2}}(\ln t + \ln x))}{2t(t^2 - \mathrm{e}^t)(\ln t - t)}\, x\Big(\frac{t}{2}\Big) x'(t) \\ & + \frac{t-2}{t(\mathrm{e}^t - t^2)} \bigg(\frac{t(\ln x + t)(x - \mathrm{e}^{-t})}{2(t - \ln t)(\frac{1}{t} - \mathrm{e}^{-t})} + \frac{\mathrm{e}^{\frac{t}{2}}(\ln x + \ln t)(x - \frac{1}{t})}{(\ln t - t)(\mathrm{e}^{-t} - \frac{1}{t})} \bigg) x\Big(\frac{t}{2}\Big) \end{aligned}$$

has solutions $x(t) = \frac{1}{t}$ and $x(t) = \mathrm{e}^{-t}$.

It should be interesting to study the relation between the decay of Kneser solutions and the asymptotic growth of the deviating arguments γ and τ.

Open problem 3. Sufficient conditions ensuring that all bounded solutions of equation

$$(a(t)\Phi(x'))' = b(t) f(x(g(t))), \quad g(t) < t, \tag{3.5}$$

are oscillatory have been given in [5, Corollary 3]. This result is a consequence of some results concerning necessary conditions for the existence of bounded nonoscillatory solutions of (3.5).

It would be interesting to obtain necessary conditions for the existence of Kneser solutions of (1.1) and, as a consequence of such result, to obtain criteria that every bounded solution of (1.1) is oscillatory.

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Asymptotic Representations of Rapid Varying Solutions of Differential Equations Asymptotically Close to the Equations with Regularly Varying Nonlinearities

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The differential equation

$$y^{(n)}=f(t,y,y',\dots,y^{(n-1)}) \tag{1}$$

is considered. Here $n\geq 2$, $f:[a,\omega[\,\times\Delta_{Y_0}\times\Delta_{Y_1}\times\cdots\times\Delta_{Y_{n-1}}\to\mathbb{R}$ is some continuous function, $-\infty<a<\omega\leq+\infty$, Y_j equals to zero, or to $\pm\infty$, Δ_{Y_j} is some one-sided neighborhood of Y_j, $j=0,1,\dots,n-1$.

The asymptotic estimations for singular, quickly varying, and Kneser solutions of equation (1) are described in the monograph by I. T. Kiguradze, T. A. Chanturia [4].

Definition 1. The solution y of equation (1), defined on the interval $[t_0,\omega[\,\subset[a,\omega[$, is called $P_\omega(Y_0,Y_1,\dots,Y_{n-1},\lambda_0)$-solution, where $-\infty\leq\lambda_0\leq+\infty$, if the next conditions take place

$$y^{(j)}(t)\in\Delta_{Y_j}\ \text{ as }\ t\in[t_0,\omega[\,,\quad \lim_{t\uparrow\omega}y^{(j)}(t)=Y_j\ \ (j=0,1,\dots,n-1),$$

$$\lim_{t\uparrow\omega}\frac{\big[y^{(n-1)}(t)\big]^2}{y^{(n-2)}(t)y^{(n)}(t)}=\lambda_0.$$

The asymptotic behavior of such solutions earlier has been investigated in the works by V. M. Evtukhov and A. M. Klopot [1–3, 5] for the differential equation

$$y^{(n)}=\sum_{i=1}^{m}\alpha_i p_i(t)\prod_{j=0}^{n-1}\varphi_{ij}(y^{(j)}),$$

where $n\geq 2$, $\alpha_i\in\{-1;1\}$, $p_i:[a,\omega[\,\to\,]0,+\infty[$ is a continuous function, $i=1,\dots,m$, $-\infty<a<\omega\leq+\infty$, $\varphi_{ij}:\Delta_{Y_j}\to\,]0,+\infty[$ is a continuous regularly varying as $y^{(j)}\to Y_j$ function of order σ_j, $j=0,1,\dots,n-1$ $(i=1,\dots,m)$.

The aim of the paper is in establishing the necessary and sufficient conditions of the existence of $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solutions of equation (1) and in finding the asymptotic representations of such solutions and their derivatives to the order $n-1$ including.

Every $P_\omega(Y_0,Y_1,\dots,Y_{n-1},1)$-solution of the differential equation (1) has (see, for example, [1]) the next a priori asymptotic properties

$$\frac{y'(t)}{y(t)}\sim\frac{y''(t)}{y'(t)}\sim\cdots\sim\frac{y^{(n)}(t)}{y^{(n-1)}(t)}\ \text{ as }\ t\uparrow\omega,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=\pm\infty,$$

where

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty.\end{cases}$$

Definition 2. The function f in the differential equation (1) is called a function, that satisfies the condition $(RN)_1$, if there exist a number $\alpha_0 \in \{-1,1\}$, a continuous function $p : [a,\omega[\, \to\,]0,+\infty[$ and continuous regularly varying as $z \to Y_j$ $(j=\overline{0,n-1})$ functions $\varphi_j : \Delta_{Y_j} \to\,]0,+\infty[$ $(j=\overline{0,n-1})$ of orders σ_j $(j=\overline{0,n-1})$, such that for all continuously differentiable functions $z_j : [a,\omega[\, \to \Delta_{Y_j}$ $(j=\overline{0,n-1})$, satisfying the conditions

$$\lim_{t\uparrow\omega} z_j(t)=Y_j, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)z_j'(t)}{z_j(t)}=\pm\infty \ \ (j=\overline{0,n-1}),$$
$$\lim_{t\uparrow\omega}\frac{z_{j-1}'(t)z_j(t)}{z_{j-1}(t)z_j'(t)}=1 \ \ (j=\overline{1,n-1}),$$

the next representation takes place

$$f(t,z_0(t),z_1(t),\dots,z_{n-1}(t))=\alpha_0 p(t)\prod_{j=0}^{n-1}\varphi_j(z_j(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Furthermore, we will use the following notations.

$$\gamma=1-\sum_{j=0}^{n-1}\sigma_j, \quad \mu_n=\sum_{j=0}^{n-2}\sigma_j(n-j-1);$$

$$\nu_j=\begin{cases} 1 & \text{if } Y_j=+\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is the right neigbourhood of zero,}\\ -1 & \text{if } Y_j=-\infty, \text{ or } Y_j=0 \text{ and } \Delta_{Y_j} \text{ is the left neigbourhood of zero} \end{cases} \quad (j=\overline{0,n-1});$$

$$J_0(t)=\int_{A_0}^t p(s)\,ds, \quad J_{00}(t)=\int_{A_{00}}^t J_0(s)\,ds,$$

where

$$A_0=\begin{cases} a & \text{if } \displaystyle\int_a^\omega p(s)\,ds=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega p(s)\,ds<+\infty, \end{cases} \qquad A_{00}=\begin{cases} a & \text{if } \displaystyle\int_a^\omega |J_0(s)|\,ds=+\infty,\\ \omega & \text{if } \displaystyle\int_a^\omega |J_0(s)|\,ds<+\infty. \end{cases}$$

Theorem 1. *Let the function f satisfy the condition $(RN)_1$ and $\gamma\neq 0$. Then for the existence of $P_\omega(Y_0,\dots,Y_{n-1},1)$-solutions of equation (1) the next conditions are necessary:*

$$\frac{p(t)}{J_0(t)}\sim\frac{J_0(t)}{J_{00}(t)} \ \ as \ \ t\uparrow\omega, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{J_0(t)}=\pm\infty,$$
$$\nu_j\lim_{t\uparrow\omega}|J_0(t)|^{\frac{1}{\gamma}}=Y_j \ \ (j=\overline{0,n-1}),$$

and, for $t\in\,]a,\omega[$, the next inequalities take place

$$\alpha_0\nu_{n-1}\gamma J_0(t)>0, \quad \nu_j\nu_{n-1}(\gamma J_0(t))^{n-j-1}>0 \ \ (j=\overline{0,n-2}).$$

As the algebraic of ρ equation

$$(1+\rho)^n=\sum_{j=0}^{n-1}\sigma_j(1+\rho)^j \tag{2}$$

has no roots with zero real part, the conditions also are sufficient for the existence of such solutions of equation (1)*. Moreover, for any such solution the next asymptotic representations*

$$y^{(j)}(t) = \Big(\frac{\gamma J_{00}(t)}{J_0(t)}\Big)^{n-j-1} y^{(n-1)}(t)[1+o(1)] \ \ (j=\overline{0,n-2}), \tag{3}$$

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=0}^{n-1} L_j\big((\frac{\gamma J_{00}(t)}{J_0(t)})^{n-j-1} y^{(n-1)}(t)\big)} = \alpha_0\nu_{n-1}\gamma J_0(t)\Big|\frac{\gamma J_{00}(t)}{J_0(t)}\Big|^{\mu_n}[1+o(1)], \tag{4}$$

take place as $t\uparrow\omega$*. Here* $L_j(y^{(j)}) = |y^{(j)}|^{-\sigma_j}\varphi_j(y^{(j)}t)$ $(j=\overline{0,n-1})$*. There exists* m*-parametric family of such solutions, if among the roots of equation* (2) *there exist* m *roots (taking into account multiply roots), the real parts of which have the sign that is among opposite to the sign* $\alpha_0\nu_{n-1}$*.*

The asymptotic representation of the $(n-1)$-th derivative of $P_\omega(Y_0,\dots,Y_{n-1},1)$-solution of equation (1) is given in the implicit form. We will indicate the conditions by implementation of which the asymptotic representations (3), (4) can be written in the explicit form.

Definition 3. The slowly varying as $y\to Y$ function $L:\Delta_Y\to]0,+\infty[$, where Y equals either zero, or $\pm\infty$, Δ_Y is a one-sided neighborhood of Y, is called satisfying the condition S_0 if the next condition takes place:

$$L(\nu e^{[1+o(1)]\ln|y|}) = L(y)[1+o(1)] \ \text{ as } \ y\to Y \ \ (y\in\Delta_Y),$$

where $\nu=\operatorname{sign} y$.

Theorem 2. *Let the conditions of Theorem 1 be satisfied and regularly varying functions* L_j $(j=\overline{0,n-1})$ *satisfy the condition* S_0*. Then for any* $P_\omega(Y_0,\dots,Y_{n-1},1)$*-solution of equation* (1) *the next asymptotic representations*

$$y^{(j)}(t) = \nu_{n-1}\Big(\frac{\gamma J_0(t)}{p(t)}\Big)^{n-j-1}\bigg|\gamma J_0(t)\Big|\frac{\gamma J_0(t)}{p(t)}\Big|^{\mu_n}\prod_{j=0}^{n-1} L_j\big(\nu_j|J_0(t)|^{\frac{1}{\gamma}}\big)\bigg|^{\frac{1}{\gamma}}[1+o(1)] \ \ (j=\overline{0,n-1})$$

take place as $t\uparrow\omega$*.*

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Optimization of the Delay Parameter for One Class of Controlled Dynamical System

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1 Mathematical model

As is known the real controlled dynamical systems contain effects with delayed action and are described by differential equations with delay in control [3]. To illustrate this, below we will consider the simplest model of marketing relation.

Let $t_1 > t_0$, $\beta > \alpha \geq 0$ and $\theta_2 > \theta_1 > 0$ be given numbers. Let market relation demand and supply be described by the functions $D(t,p)$ and $S(t,q)$, which are continuous and continuously differentiable with respect to p and q.

Let the function $p(t) \in P = [\alpha,\beta]$, $t \in I_1 = [t_0-\theta_2, t_1]$ be price of a good, changing over time. Suppose that at time $t \in I_2 = [t_0, t_1]$ will be satisfied demand of consumer which has been ordered at time $t-\theta$, i.e. when price of a good was $p(t-\theta)$. Here $\theta \in I_3 = [\theta_1, \theta_2]$ is so-called delay parameter.

The function

$$R(t) = D(t,p(t)) - S(t,p(t-\theta)), \ \ t \in I_2,$$

we call the disbalance index.

If $R(t) = 0$, then at the moment t we do not have disbalance between supply and demand, and the customer will buy exactly the quantity of goods he needs.

It is clear that at various time moment t the disbalance index $R(t)$ is possible to be not positive as well as positive. At time t, if $R(t) > 0$, then demand exaggerates supply. If $R(t) < 0$, then supply exaggerates demand. To describe development of marketing relation process in time, i.e. create dynamical model, we consider the integral index of disbalance

$$y(t) = R(t_0) + \int\limits_{t_0}^{t} R(s)\,ds. \tag{1.1}$$

The function $y(t)$ gives complete information about the disbalance from the initial time t_0 to any time t.

From (1.1) we get the differential equation

$$\dot{y}(t) = D(t,p(t)) - S(t,p(t-\theta)), \ \ t \in I_2 \tag{1.2}$$

with the initial condition

$$y(t_0) = y_0 := R(t_0).$$

2 Statement of the problem. Necessary optimality conditions

Let $O \subset \mathbb{R}^n$ be an open set and $U \subset \mathbb{R}^r$ be a convex and compact set. Let the $(n+1)$-dimensional function

$$F(t,x,u,v) = \big(f^0(t,x,u,v), f(t,x,u,v)\big)^\top,$$

where $f = (f^1,\dots,f^n)^\top$, be continuous on $I_2 \times O \times U^2$ and continuously differentiable with respect to x and u, v. Furthermore, let $x_0, x_1 \in O$ be fixed points and let Ω be a set of absolutely continuous control functions $u(t) \in U$, $t \in I_1$. To each element $w = (\theta, u) \in \Lambda := I_3 \times \Omega$ we assign the differential equation

$$\dot{x}(t) = f\big(t, x(t), u(t), u(t-\theta)\big), \quad t \in (t_0, t_1) \tag{2.1}$$

with the initial condition

$$x(t_0) = x_0. \tag{2.2}$$

Definition 2.1. Let $w = (\theta, u) \in \Lambda$. A function $x(t) = x(t; w) \in O$, $t \in I_2$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element w and defined on the interval I_2 if it satisfies condition (2.2) and is continuously differentiable and satisfies equation (2.1) everywhere on (t_0, t_1).

Definition 2.2. An element $w = (\theta, u) \in \Lambda$ is said to be admissible if the corresponding solution $x(t) = x(t; w)$ satisfies the condition

$$x(t_1) = x_1. \tag{2.3}$$

Denote by Λ_0 the set of admissible elements.

Definition 2.3. An element $w_0 = (\theta_0, u_0) \in \Lambda_0$ is said to be optimal if for an arbitrary element $w \in \Lambda_0$ we have

$$J(w_0) \le J(w), \tag{2.4}$$

where

$$J(w) = \int_{t_0}^{t_1} f^0(t, x(t), u(t), u(t-\theta))\, dt$$

and $x(t) = x(t; w)$.

(2.1)–(2.4) is called the optimization problem of delay parameter θ and control $u(t)$.

Theorem 2.1. *Let w_0 be an optimal element and let $x_0(t) = x(t; w_0)$ be the optimal trajectory. Then there exists a nontrivial solution $\Psi(t) = (\psi_0(t), \psi(t))$ of the equation*

$$\dot{\psi}(t) = -\Psi(t) F_x[t], \tag{2.5}$$

where

$$F_x[t] = F_x\big(t, x_0(t), u_0(t), u_0(t-\theta_0)\big),$$

such that $\psi_0(t) \equiv const \le 0$ and the following conditions hold:

(i_1) *the integral condition for the optimal delay parameter θ_0*

$$\int_{t_0}^{t_1} \Psi(t) F_v[t] \dot{u}_0(t-\theta_0)\, dt = 0;$$

(i_2) *the integral maximum principle for the optimal control* $u_0(t)$

$$\int\limits_{t_0}^{t_1} \Psi(t)\big[F_u[t]u_0(t)+F_v[t]u_0(t-\theta_0)\big]\,dt=\max_{u(t)\in\Omega}\int\limits_{t_0}^{t_1} \Psi(t)\big[F_u[t]u(t)+F_v[t]u(t-\theta_0)\big]\,dt.$$

The necessary optimality condition for the delay parameter in controls for the optimization problem with the Meyer type functional is provided in [2].

3 Optimization problem for equation (1.2). Necessary optimality conditions

Let y_1 be a fixed number and let V be a set of absolutely continuous control functions $p(t)\in P$, $t\in I_1$. To each element $\vartheta=(\theta,p)\in\Pi:=I_3\times V$ we assign the differential equation

$$\dot y=D(t,p(t))-S(t,p(t-\theta)),\ \ t\in I_2$$

with the initial condition

$$y(t_0)=y_0.$$

Definition 3.1. An element $\vartheta=(\theta,p)\in\Pi$ is said to be admissible if the corresponding solution $y(t)=y(t;\vartheta)$ satisfies the condition

$$y(t_1)=y_1.$$

Denote by Π_0 the set of admissible elements.

Definition 3.2. An element $\vartheta_0=(\theta_0,p_0)\in\Pi_0$ is said to be optimal if for an arbitrary element $\vartheta\in\Pi_0$ we have

$$\int\limits_{t_0}^{t_1} g(t,p_0(t))\,dt\le\int\limits_{t_0}^{t_1} g(t,p(t))\,dt,$$

where the function $g(t,p)$ is continuous and continuously differentiable with respect to p.

It is clear that for the considered problem we have $\dot\psi=0$ (see (2.5)). Taking into account the last equation from Theorem 2.1 it follows

Theorem 3.1. *Let ϑ_0 be an optimal element. Then there exists a nontrivial vector $\Psi=(\psi_0,\psi)$, $\psi_0\le 0$ such that the following conditions hold:*

(i_3) *the integral condition for the optimal delay parameter* θ_0

$$\psi\int\limits_{t_0}^{t_1} S_q(t,p_0(t-\theta_0))\dot p_0(t-\theta_0)\,dt=0;$$

(i_4) *the integral maximum principle for the optimal control* $p_0(t)$

$$\int\limits_{t_0}^{t_1}\Big[\big(\psi_0 g_p(t,p_0(t))+\psi D_p(t,p_0(t))\big)p_0(t)-\psi S_q(t,p_0(t-\theta_0))p_0(t-\theta_0)\Big]\,dt$$

$$\max_{p(t)\in V}\int\limits_{t_0}^{t_1}\Big[\big(\psi_0 g_p(t,p_0(t))+\psi D_p(t,p_0(t))\big)p(t)-\psi S_q(t,p_0(t-\theta_0))p(t-\theta_0)\Big]\,dt.$$

Analogous problem for equation (1.2) with the fixed θ is investigated in [1].

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Asymptotic of Rapid Varying Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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We consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0,\\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ and } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

It means that in the considered equation the continuous function φ and its first order derivative are [6, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92] rapidly varying as $y \to Y_0$.

For two-term differential equations of second order with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works by M. Marić [6], V. M. Evtukhov and his students N. G. Drik, A. G. Chernikova [2,3].

In the works by V. M. Evtukhov, A. G. Chernikova [3] for the differential equation (1) of second order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0,\lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$.

In the works by V. M. Evtukhov, N. V. Sharay [5] for the differential equation (1) of third order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0,\lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R} \setminus \{0,1,\frac{1}{2}\}$. In this work, we propose the distribution of [3] results to third-order differential equations.

Solution y of the differential equation (1) specified on the interval $[t_0,\omega[\subset [a,\omega[$ is said to be $P_\omega(Y_0,\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either} & 0,\\ \text{or} & \pm\infty, \end{cases} \quad k=1,2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1) in the non-singular case when $\lambda_0\in\mathbb{R}\setminus\{0,1,\frac12\}$, as well as asymptotic representations as $t\uparrow\omega$ for such solutions and their derivatives up to the second order inclusively.

Without loss of generality, we assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is the left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is the right neighborhood of } Y_0,\end{cases} \tag{3}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$, when $Y_0=0$ and $y_0>1$ $(y_0<-1)$, when $Y_0=+\infty$ (when $Y_0=-\infty$).

A function $\varphi:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), belongs to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the work [3] which extends the class of function Γ, introduced by L. Khan (see, for example, [1, Ch. 3, § 3.10, p. 175]). Using properties from this class the main results are obtained.

We introduce the necessary auxiliary notation. We assume that the domain of the function $\varphi\in\Gamma_{Y_0}(Z_0)$ is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$\pi_\omega(t)=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\qquad J_1(t)=\int\limits_{A_1}^{t}p^{\frac13}(\tau)\,d\tau,\quad \Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)},$$

where

$$A_1=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=const,\\ a & \text{if } \displaystyle\int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=\pm\infty,\end{cases}\qquad B_1=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=const,\\ y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=\pm\infty.\end{cases}$$

Considering the definition of $P_\omega(Y_0,1)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,1)$-solution and of its first derivative in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0,\ \text{ if } Y_0=0,\quad \nu_0\nu_1>0,\ \text{ if } Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$ as $y\to Y_0$ and increases by Δ_{Y_0}, because on this interval $\Phi_1'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi_1^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ_1^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi_1(y)=\begin{cases}\text{eitherr} & 0,\\ \text{or} & +\infty,\end{cases}\qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{or } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{or } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\Phi_1(y_0).$$

In addition to the indicated notation, using Φ_1^{-1} we also introduce the auxiliary functions

$$q_1(t) = \frac{\alpha_0 \nu_1 J_1(t)}{p^{\frac{1}{3}}(t)\varphi^{\frac{1}{3}}(\Phi_1^{-1}(\nu_1 J_1(t)))(\Phi_1^{-1}(\nu_1 J_1(t)))^{\frac{2}{3}}},$$
$$H_1(t) = \frac{\Phi_1^{-1}(\nu_1 J_1(t))\varphi'(\Phi_1^{-1}(\nu_1 J_1(t)))}{\varphi(\Phi_1^{-1}(\nu_1 J_1(t)))},$$
$$J_2(t) = \int_{A_2}^{t} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau, \quad J_3(t) = \int_{A_3}^{t} J_2(\tau)\,d\tau,$$

where

$$A_2 = \begin{cases} t_0 & \text{if } \displaystyle\int_{t_2}^{\omega} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_{a}^{\omega} p(\tau)\ \varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau < +\infty, \end{cases}$$
$$A_3 = \begin{cases} t_0 & \text{if } \displaystyle\int_{t_3}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_{a}^{\omega} J_2(\tau)\,d\tau < +\infty, \end{cases} \qquad t_2, t_3 \in [a, \omega).$$

For equation (1) the following assertions are valid.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$. For the existence of $P_\omega(Y_0, 1)$-solutions of the differential equation* (1)*, it is necessary to comply with the conditions*

$$\alpha_0 \nu_0 > 0, \quad \mu_0 \nu_1 J_1(t) > 0 \textit{for } \ t \in (a, \omega);$$
$$\nu_1 \lim_{t \uparrow \omega} J_1(t) = Z_0, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_1'(t)}{J_1(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q(t) = 1$$

and

$$\lim_{t \uparrow \omega} \frac{p(t)\varphi\big(\Phi_1^{-1}(\nu_1 J_1(t))\big) J_3(t)}{(J_2(t))^2} = 1.$$

Moreover, for each such solution there take place the asymptotic representations

$$y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0 - 1) J_1(t)\big)\Big[1 + \frac{o(1)}{H_1(t)}\Big] \ \textit{ as } \ t \uparrow \omega,$$
$$y'(t) = \nu_1 p^{\frac{1}{3}}(t)\varphi^{\frac{1}{3}}\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)^{\frac{2}{3}}[1 + o(1)] \ \textit{ as } \ t \uparrow \omega,$$
$$y''(t) = \alpha_0 J_2(t)[1 + o(1)] \ \textit{ as } \ t \uparrow \omega.$$

In addition, sufficient conditions for the existence of such solutions are obtained.

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On Below Estimates for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non-negative values such that the following integral condition hold

$$\int\limits_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \gamma \neq 0, \tag{1.3}$$

$$\int\limits_0^1 x(1-x)Q(x)\,dx < \infty. \tag{1.4}$$

A function y is a *solution to problem* (1.1), eqrefTelnova eq 2 if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

For $\gamma < 0$, $\alpha \leq 2\gamma - 1$, $-\infty < \beta < +\infty$ or $\gamma < 0$, $\beta \leq 2\gamma - 1$, $-\infty < \alpha < +\infty$, the set $T_{\alpha,\beta,\gamma}$ is empty, the first eigenvalue of problem $(1.1), (1.2)$ does not exist. Given $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ or $\gamma > 0$, $-\infty < \alpha$, $\beta < +\infty$, $Q \in T_{\alpha,\beta,\gamma}$, we obtain

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y], \quad \text{where} \quad R[Q,y] = \frac{\int\limits_0^1 y'^2\,dx - \int\limits_0^1 Q(x)y^2\,dx}{\int\limits_0^1 y^2\,dx}\,.$$

For any α, β, γ, $\gamma \neq 0$, for any $Q \in T_{\alpha,\beta,\gamma}$, the following relations hold

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_0^1(0,1)} R[Q,y] \leq \inf_{y \in H_0^1(0,1)\setminus\{0\}} \frac{\int\limits_0^1 y'^2\,dx}{\int\limits_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1.

1. *If* $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ *or* $0 < \gamma < 1$, *then* $m_{\alpha,\beta,\gamma} = -\infty$.

2.1. *If* $\gamma = 1$, $\alpha, \beta \leqslant 0$, *then* $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.

2.2. *If* $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ *or* $\alpha \leqslant 0 < \beta \leqslant 1$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.3. *If* $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.4. *If* $\gamma > 1$, $\alpha, \beta \leqslant 0$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

Proof. By the Hölder inequality, for any $y \in H_0^1(0,1)$, for any $x \in (0,1)$, we have

$$y^2(x) = \bigg(\int\limits_0^x y'(t)\,dt \bigg)^2 \leqslant x \int\limits_0^x {y'}^2(t)\,dt, \tag{2.1}$$

$$y^2(x) = \bigg(-\int\limits_x^1 y'(t)\,dt \bigg)^2 \leqslant (1-x) \int\limits_x^1 {y'}^2(t)\,dt.$$

Then

$$\frac{y^2}{x(1-x)} = \frac{y^2}{x} + \frac{y^2}{1-x} \leqslant \int\limits_0^x {y'}^2(t)\,dt + \int\limits_x^1 {y'}^2(t)\,dt = \int\limits_0^1 {y'}^2(t)\,dt,$$

$$y^2(x) \leqslant x(1-x) \int\limits_0^1 {y'}^2(t)\,dt. \tag{2.2}$$

1.1. If $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$, then there exists a number $r > 0$ such that $\alpha > 2\gamma - 1 + r$, $\beta > 2\gamma - 1 + r$. For $0 < \varepsilon < 1$, consider the function $Q_\varepsilon \in T_{\alpha,\beta,\gamma}$:

$$Q_\varepsilon(x) = \begin{cases} r^{\frac{1}{\gamma}} (1-\varepsilon)^{\frac{1}{\gamma}} \varepsilon^{-\frac{r}{\gamma}} x^{-\frac{\alpha+1-r}{\gamma}} (1-x)^{-\frac{\beta}{\gamma}}, & 0 < x \leqslant \varepsilon; \\ r^{\frac{1}{\gamma}} (1-\varepsilon)^{-\frac{r}{\gamma}} \varepsilon^{\frac{1}{\gamma}} x^{-\frac{\alpha}{\gamma}} (1-x)^{-\frac{\beta+1-r}{\gamma}}, & \varepsilon < x < 1. \end{cases}$$

By the Hölder inequality, for any function $y \in H_0^1(0,1)$, we have

$$\int\limits_0^1 y^2\,dx = \int\limits_0^\varepsilon y^2\,dx + \int\limits_\varepsilon^1 y^2\,dx \leqslant \frac{\varepsilon^2}{2} \int\limits_0^\varepsilon {y'}^2\,dx + r^{-\frac{1}{\gamma}} \varepsilon^{-\frac{1}{\gamma}} (1-\varepsilon)^{\frac{r}{\gamma}} \int\limits_\varepsilon^1 Q_\varepsilon(x) y^2\,dx.$$

Then

$$\int\limits_0^1 Q_\varepsilon(x) y^2\,dx \geqslant \int\limits_\varepsilon^1 Q_\varepsilon(x) y^2\,dx \geqslant r^{\frac{1}{\gamma}} \varepsilon^{\frac{1}{\gamma}} (1-\varepsilon)^{-\frac{r}{\gamma}} \bigg(\int\limits_0^1 y^2\,dx - \frac{\varepsilon^2}{2} \int\limits_0^1 {y'}^2\,dx \bigg).$$

For any function $y_* \in H_0^1(0,1)$, for example, for $y_* = \sin \pi x$,

$$R[Q_\varepsilon, y_*] \leqslant \frac{\int_0^1 {y_*'}^2\,dx + r^{\frac{1}{\gamma}} \varepsilon^{\frac{1}{\gamma}} (1-\varepsilon)^{-\frac{r}{\gamma}} \big(\frac{\varepsilon^2}{2} \int_0^1 {y_*'}^2\,dx - \int_0^1 {y_*}^2\,dx \big)}{\int_0^1 {y_*}^2\,dx}.$$

Therefore,

$$\inf_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y]\leqslant \lim_{\varepsilon\to 0}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q_\varepsilon,y]\leqslant \lim_{\varepsilon\to 0} R[Q_\varepsilon,y_*]=-\infty.$$

1.2. Let $0<\gamma<1$ and α,β be arbitrary real numbers. For $0<\varepsilon<1$, consider the function $Q_\varepsilon\in T_{\alpha,\beta,\gamma}$:

$$Q_\varepsilon(x)=\begin{cases} 0, & 0\leqslant x<\dfrac{1}{2}-\dfrac{\varepsilon}{2},\ \ \dfrac{1}{2}+\dfrac{\varepsilon}{2}<x\leqslant 1;\\[2mm] \varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & \dfrac{1}{2}-\dfrac{\varepsilon}{2}\leqslant x\leqslant \dfrac{1}{2}+\dfrac{\varepsilon}{2}\,. \end{cases}$$

If $y_*=\sin\pi x$ and $C_\varepsilon=\min_{[\frac{1}{2}-\frac{\varepsilon}{2},\frac{1}{2}+\frac{\varepsilon}{2}]} x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}$, then

$$\int_0^1 Q_\varepsilon(x){y_*}^2\,dx=\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}}\varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}\sin^2\pi x\,dx$$

$$\geqslant C_\varepsilon\cdot\varepsilon^{-\frac{1}{\gamma}}\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}}\frac{1-\cos 2\pi x}{2}\,dx=C_\varepsilon\cdot\varepsilon^{-\frac{1}{\gamma}}\Big(\frac{\varepsilon}{2}+\frac{\sin\pi\varepsilon}{2\pi}\Big).$$

Similarly to case 1.1, we obtain $m_{\alpha,\beta,\gamma}=-\infty$.

2.1. Let $\gamma=1$ and $\alpha,\beta\leqslant 0$. It is known (see, for ex., [1]) that for any $y\in H_0^1(0,1)$, the inequality

$$\sup_{[0,1]} y^2\leqslant \frac{1}{4}\int_0^1 {y'}^2\,dx$$

holds. For any functions $Q\in T_{\alpha,\beta,\gamma}$ and $y\in H_0^1(0,1)$, we obtain

$$\int_0^1 Q(x)y^2\,dx\leqslant \sup_{[0,1]} y^2\int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx\leqslant \sup_{[0,1]} y^2\leqslant \frac{1}{4}\int_0^1 {y'}^2\,dx.$$

Therefore,

$$m_{\alpha,\beta,\gamma}\geqslant \frac{3}{4}\inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 {y'}^2\,dx}{\int_0^1 y^2\,dx}=\frac{3}{4}\,\pi^2.$$

2.2. Let $\gamma=1$, $\beta\leqslant 0<\alpha\leqslant 1$. In virtue of (2.1), for any function $Q\in T_{\alpha,\beta,\gamma}$, we have

$$\int_0^1 Q(x)y^2\,dx\leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx\leqslant \sup_{[0,1]}\frac{y^2}{x}\leqslant \int_0^1 {y'}^2\,dx.$$

Then

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 {y'}^2-\int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\geqslant 0.$$

The case $\alpha \leqslant 0 < \beta \leqslant 1 = \gamma$ is symmetrical to the case $\beta \leqslant 0 < \alpha \leqslant 1 = \gamma$.

2.3. Let $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$. In virtue of (2.2),

$$\int_0^1 Q(x)y^2\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta} \int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x(1-x)} \leqslant \int_0^1 y'^2\,dx.$$

and also $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.4. Let $\gamma > 1$, $\alpha, \beta \leqslant 0$. By the Hölder inequality, for any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_0^1(0,1)$, we obtain the same result due to

$$\int_0^1 Q(x)y^2\,dx \leqslant \Big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \Big)^{\frac{\gamma-1}{\gamma}} \leqslant \Big(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}}\,dx \Big)^{\frac{\gamma-1}{\gamma}} \leqslant \int_0^1 y'^2\,dx. \tag{2.3}$$

3 On precise estimates for $m_{\alpha,\beta,\gamma}$ as $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$

Theorem 3.1. *If $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $m_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}} \tag{3.1}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{3.2}$$

Proof. Let $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$. In virtue of (2.3), for any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_0^1(0,1)$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y] \geqslant \inf_{y \in H_0^1(0,1)\setminus\{0\}} G[y] = m,$$

where

$$G[y] = \frac{\int_0^1 y'^2\,dx - \big(\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \big)^{\frac{\gamma-1}{\gamma}}}{\int_0^1 y^2\,dx},$$

and

$$m_{\alpha,\beta,\gamma} \geqslant m.$$

Following the proof of Theorem 2.1 [2], we obtain that the minimizing sequence of $G[y]$ converges in $H_0^1(0,1)$ to some function u and

$$\inf_{y \in H_0^1(0,1)\setminus\{0\}} G[y] = G[u] = m.$$

Similarly, the function u satisfies equation (3.1) and the integral condition (3.2). Since u is non-negative on $(0,1)$, the graph of u cannot cross the axis Ox. The touching the axis Ox is also impossible due to the existence and uniqueness theorem for the solution of the Cauchy problem, as $\gamma > 1$ and $\frac{\gamma+1}{\gamma-1} > 1$. Therefore, the function u is positive on $(0,1)$.

On $(0,1)$ the function $Q_*(x)=x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}$ satisfies conditions (1.3) and (1.4). Since for $Q=Q_*$ and $\lambda=m$ the function u satisfies equation (1.1), satisfies conditions (1.2), since u is continuous on $[0,1]$, positive on $(0,1)$ and its derivative u' is continuous on $(0,1)$, the function u is the first eigenfunction of problem (1.1)–(1.4) with $Q=Q_*$ and the first eigenvalue $\lambda_1(Q_*)=m$.

Then

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\inf_{y\in H_0^1(0,1)\setminus\{0\}}R[Q,y]\leqslant\inf_{y\in H_0^1(0,1)\setminus\{0\}}R[Q_*,y]=R[Q_*,u]=G[u]=m.$$

Therefore, we obtain $m_{\alpha,\beta,\gamma}=m$.

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Existence and Multiplicity of Periodic Solutions to Second-Order Differential Equations with Attractive Singularities

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Consider a second-order ordinary differential equation of the form

$$u'' + \frac{g(t)}{u^\lambda} = h(t)u^\delta + \mu f(t), \tag{1}$$

where $g, h, f \in L(\mathbb{R}/T\mathbb{Z})$, $g(t) \geq 0$ for a.e. $t \in \mathbb{R}$, $\overline{g} > 0$, $\overline{h} < 0$, $\overline{f} > 0$, $\lambda > 0$, $\delta \in (0,1)$, and $\mu \geq 0$ is a parameter.

Throughout we use the following notation.

- $C(\mathbb{R}/T\mathbb{Z})$ is a Banach space of T-periodic continuous functions $u : \mathbb{R} \to \mathbb{R}$ endowed with a norm $\|u\|_C = \max\{|u(t)| : \ t \in [0,T]\}$.
- $AC^1(\mathbb{R}/T\mathbb{Z})$ is a set of T-periodic functions $u : \mathbb{R} \to \mathbb{R}$ such that u and u' are absolutely continuous.
- $L^p(\mathbb{R}/T\mathbb{Z})$ $(p \geq 1)$ is a Banach space of T-periodic functions $h : \mathbb{R} \to \mathbb{R}$ that are integrable with the p-th power on the interval $[0,T]$ endowed with a norm

  $$\|h\|_p = \bigg(\int\limits_0^T |h(s)|^p \, ds\bigg)^{1/p}.$$

  We write $L(\mathbb{R}/T\mathbb{Z})$ instead of $L^1(\mathbb{R}/T\mathbb{Z})$.
- $[x]_+ = \frac{1}{2}\,(|x| + x)$, $[x]_- = \frac{1}{2}\,(|x| - x)$.
- If $h \in L(\mathbb{R}/T\mathbb{Z})$ then $\overline{h} = \frac{1}{T} \int\limits_0^T h(s)\, ds$.

By a T-periodic solution to (1) we understand a function $u \in AC^1(\mathbb{R}/T\mathbb{Z})$ which is positive and satisfies the equality (1) for almost every $t \in \mathbb{R}$.

Theorem 1. *Let $[h]_+, [f]_+ \in L^p(\mathbb{R}/T\mathbb{Z})$ with $p \geq 1$. Let, moreover, there exist $\varphi \in L^q(\mathbb{R}/T\mathbb{Z})$ $(q \geq 1)$ such that*[1]

$$[h]_+(t) + [f]_+(t) \leq \varphi(t) g^{\frac{q-1}{q}}(t) \ \text{ for a.e. } \ t \in \mathbb{R}$$

and let

$$\lim_{x \to t_+} \int\limits_x^{t+T/2} \frac{g(s)}{(s-t)^{\frac{\lambda(2p-1)q}{p}}}\, ds + \lim_{x \to t_-} \int\limits_{t+T/2}^{x+T} \frac{g(s)}{(t+T-s)^{\frac{\lambda(2p-1)q}{p}}}\, ds = +\infty$$

be fulfilled for every $t \in \mathbb{R}$. Then there exist $\mu^* \geq \mu_* > 0$ such that

- *Eq. (1) has at least two T-periodic solutions provided $\mu > \mu^*$;*
- *Eq. (1) has at least one T-periodic solution provided $\mu = \mu^*$;*
- *Eq. (1) has no T-periodic solution provided $\mu \in [0, \mu_*)$.*

Remark. In the case when $h(t) \leq 0$ for a. e. $t \in \mathbb{R}$ it can be proved that the numbers μ^* and μ_* appearing in Theorem 1 coincide.

Before we pass to the proof of Theorem 1, we introduce some definitions and notation.

Definition 1. We say that $\alpha, \beta \in AC^1(\mathbb{R}/T\mathbb{Z})$ are, respectively, lower and upper functions to the T-periodic problem for (1), if they are positive and

$$\alpha''(t) + \frac{g(t)}{\alpha^\lambda(t)} \geq h(t)\alpha^\delta(t) + \mu f(t) \ \text{ for a.e. } \ t \in \mathbb{R},$$

resp.

$$\beta''(t) + \frac{g(t)}{\beta^\lambda(t)} \leq h(t)\beta^\delta(t) + \mu f(t) \ \text{ for a.e. } \ t \in \mathbb{R}.$$

Definition 2. We say that a lower function α and an upper function β to the T-periodic problem for (1) are well-ordered if

$$\alpha(t) \leq \beta(t) \ \text{ for } \ t \in \mathbb{R}.$$

Definition 3. We say that a lower function α, resp. an upper function β to the T-periodic problem for (1) is strict if the inequality

$$\alpha(t) \leq u(t), \ \text{ resp. } \ u(t) \leq \beta(t) \ \text{ for } \ t \in \mathbb{R}$$

implies

$$\alpha(t) < u(t), \ \text{ resp. } \ u(t) < \beta(t) \ \text{ for } \ t \in \mathbb{R}$$

provided u is a T-periodic solution to (1).

Notation. We will write $\alpha(t;\mu)$, $\beta(t;\mu)$, or $u(t;\mu)$ to emphasize that the lower function α, the upper function β, or the solution u to the T-periodic problem for (1) corresponds to the particular parameter μ.

Sketch of the proof of Theorem 1. First we show that every T-periodic solution u to (1) is bounded from above. In particular, the following assertion holds.

[1] If $q = 1$ then we put $g^{\frac{q-1}{q}}(t) = 1$ for $t \in \mathbb{R}$.

Lemma 1. *There exists a non-decreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $\mu > 0$ we have*

$$u(t;\mu) < \rho(\mu)$$

provided u is a T-periodic solution to (1).

A condition $\delta > 0$ is essential in the proof of Lemma 1. The next step is a construction of well-ordered strict lower and upper functions to the T-periodic problem for (1).

Lemma 2. *Let the assumptions of Theorem 1 be fulfilled. Then for every $\mu > 0$ there exists a strict lower function α to the T-periodic problem for* (1)*. Moreover,*

$$\alpha(t;\mu) < u(t;\mu) \ \ \textit{for} \ \ t \in \mathbb{R}, \ \ \mu > 0$$

whenever u is a T-periodic solution to (1).

An important property of the lower functions $\alpha(t;\mu)$ appearing in Lemma 2 is that they are constructed in such a way that

$$\alpha(t;\mu_1) \leq \alpha(t;\mu_2) \ \ \text{for} \ \ t \in \mathbb{R} \ \ \text{whenever} \ \ \mu_1 \geq \mu_2.$$

Lemma 3. *For every μ sufficiently large there exists a strict upper function β to the T-periodic problem for* (1) *such that*

$$\alpha(t;\mu) < \beta(t;\mu) < \rho(\mu) \ \ \textit{for} \ \ t \in \mathbb{R},$$

where ρ, resp. α are functions appearing in Lemma 1, resp. Lemma 2.

Now the condition $\delta < 1$ is essential in construction of the upper functions β in Lemma 3.

The next step is obvious – for sufficiently large μ we have constructed well-ordered lower and upper functions α and β. Therefore there exists at least one T-periodic solution u to (1) between them. Moreover, since α and β are strict, we have

$$\alpha(t;\mu) < u(t;\mu) < \beta(t;\mu) \ \ \text{for} \ \ t \in \mathbb{R}, \ \ \mu \text{ sufficiently large.}$$

Furthermore, if we rewrite T-periodic problem for (1) in an equivalent operator form

$$u = M_\mu[u]$$

then it follows that the Leray-Schauder degree of the operator $I - M_\mu$ over the set

$$\Omega_\mu \stackrel{def}{=} \big\{x \in C(\mathbb{R}/T\mathbb{Z}) : \ \alpha(t;\mu) < x(t) < \beta(t;\mu) \text{ for } t \in \mathbb{R}\big\}$$

is different from zero. More precisley,

$$d_{LS}(I - M_\mu, \Omega_\mu, 0) = 1 \ \ \text{for} \ \ \mu \text{ sufficiently large.} \tag{2}$$

Thus we have proved the existence of at least one T-periodic solution to (1) in Ω_μ (for every μ sufficiently large), and have established the relation (2).

On the other hand, the following assertion holds.

Lemma 4. *Let the assumptions of Theorem 1 be fulfilled. Then there exists $\mu_* > 0$ such that there is no T-periodic solution to* (1) *with $\mu \in [0, \mu_*)$.*

For every $\mu > 0$ we define a set

$$\Psi_\mu \overset{def}{=} \big\{x \in C(\mathbb{R}/T\mathbb{Z}):\ \alpha(t;\mu) < x(t) < \rho(\mu) \text{ for } t \in \mathbb{R}\big\}.$$

Let μ_0 be arbitrary but fixed and let, moreover, it be sufficiently large such that

$$d_{LS}(I - M_{\mu_0}, \Omega_{\mu_0}, 0) = 1.$$

Then, according to Lemma 4 we have

$$d_{LS}(I - M_\mu, \Psi_{\mu_0}, 0) = 0 \ \text{ for } \ \mu \in [0, \mu_*).$$

Furthermore, due to the fact that ρ is non-decreasing and α is non-increasing with respect to μ, from Lemmas 1 and 2 it follows that there is no T-periodic solution to (1) on $\partial\Psi_{\mu_0}$ for $\mu \in [\mu_*, \mu_0]$. Consequently,

$$d_{LS}(I - M_{\mu_0}, \Psi_{\mu_0}, 0) = 0.$$

Now, in view of Lemma 3 we have $\Omega_{\mu_0} \subsetneq \Psi_{\mu_0}$, and so the additive property of the Leray-Schauder degree results in

$$d_{LS}(I - M_{\mu_0}, \Psi_{\mu_0} \setminus \Omega_{\mu_0}, 0) = -1,$$

i.e., there is another T-periodic solution to (1) in $\Psi_{\mu_0} \setminus \Omega_{\mu_0}$.

Now define

$$A \overset{def}{=} \big\{\tau > 0:\ \text{Eq. (1) has at least two } T\text{-periodic solutions for every } \mu \geq \tau\big\}.$$

Obviously, on account of the above-proven, the set A is nonempty. Moreover, according to Lemma 4, the set A is bounded from below by μ_*. Put

$$\mu^* \overset{def}{=} \inf A,$$

and let $\{\mu_n\}_{n=1}^{+\infty}$ be a sequence of parameters such that

$$\mu_n > \mu^* \ \text{ and } \lim_{n\to+\infty} \mu_n = \mu^*.$$

Obviously, there exist a sequence of T-periodic solutions $\{u(\,\cdot\,;\mu_n)\}_{n=1}^{+\infty}$ to (1) (with $\mu = \mu_n$). In addition, with respect to Lemmas 1 and 2, this sequence of solutions is uniformly bounded and equicontinuous. Thus, by standard arguments one can prove that there exists also at least one T-periodic solution to (1) with $\mu = \mu^*$. Now the sketch of the proof of Theorem 1 is complete.

On the Detection of Exact Number of Limit Cycles for Autonomous Systems on the Cylinder

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Consider the planar autonomous differential system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where the functions $P, Q : R^2 \to R$ are 2π-periodic in the first variable. Under this assumption we can identify the phase space of (1) with the cylinder $Z := S^1 \times R$, where S^1 is the unit circle. The most difficult problem in the qualitative investigation of autonomous differential systems is the localization and the estimate of the number of limit cycles.

In the case of a cylindrical phase space we have to distinguish two kinds of limit cycles. A limit cycle of system (1) which does not surround Z is called a limit cycle of the first kind, otherwise it is called a limit cycle of the second kind. Whereas the existence of a limit cycle of the first kind of system (1) requires the existence of an equilibrium point, a limit cycle of the second kind can exist without the existence of any equilibrium point [1, p. 34–35], [2, p. 218–227]. For the study of limit cycles of the first kind, the methods for planar autonomous systems can be applied (see, e.g. [2]). In particular, a well-known way to get an upper bound for the number of limit cycles of the first kind in planar systems is to check whether the criteria of I. Bendixson and H. Dulac [2] can be applied.

The method of the Dulac function has been extended by L. Cherkas [3]. The type of functions he has introduced nowadays is called Dulac–Cherkas function [7]. The existence of a Dulac–Cherkas function has the following advantages over a Dulac function: it guarantees that all limit cycles are hyperbolic (there is no multiple limit cycle), it provides some annuli containing a unique limit cycle (approximate localization of a limit cycle), it yields a simple criterion to determine the stability of limit cycles and provides lower and upper bounds for their maximum number. These functions have been applied by L. Cherkas and his coauthors also for the investigation of limit cycles of the second kind [4, 5, 8].

The fundamental importance of a Dulac–Cherkas function consists in the fact that its zero-level set defines curves which are crossed transversally by the trajectories of the corresponding system. We denote these curves in what follows as transversal curves. By this way, the cylindrical phase space is divided into doubly connected regions, where we have to distinguish between interior regions whose boundaries consist of transversal curves and which contain a unique limit cycle, and two outer regions, where only one boundary of these regions is a transversal curve and which contain at most one limit cycle. To be able to determine the exact number of limit cycles we have to investigate the existence of a limit cycle in the two outer regions. The main contribution of this paper is to show that the existence of a unique limit cycle in the outer regions can be established either by means of the existence of additional Dulac–Cherkas functions or by factorized Dulac functions. Thus, we present results on the exact number of limit cycles of the second kind.

The estimate of the number of limit cycles in some given region depends also on the structure of the region itself. Hence, our first assumption reads

(A_0). Let G be an open bounded doubly connected region on Z whose boundary consists of two simple closed curves Δ_u and Δ_l surrounding Z. We suppose that Δ_u is located above Δ_l, that is, Δ_u is the upper boundary and Δ_l is the lower boundary of G.

We denote by $C^1_{2\pi}(G,R)$ the space of continuously differentiable functions mapping G into R and which are 2π-periodic in the first variable. For the following we assume:

(A_1). The functions P and Q belong to the space $C^1_{2\pi}(G,R)$.

(A_2). G does not contain an equilibrium point of (1).

Assumption (A_2) implies that any closed orbit of system (1) completely located in G must surround the cylinder Z. That means that any limit cycle of system (1) in G is a limit cycle of the second kind which we denote by Γ. Our goal is to determine or at least to estimate the number of limit cycles of the second kind of system (1) in G. We denote this number by $\sharp\,\Gamma(G)$. The vector field defined by system (1) is denoted by X.

A known tool to estimate the number $\sharp\,\Gamma(G)$ is the Dulac function.

Definition 1. A function $D\in C^1_{2\pi}(G,R)$ is called a Dulac function of system (1) in G if $\operatorname{div}(DX)$ does not change sign in G.

The following result is well-known [2].

Theorem 1. *Suppose the assumptions* (A_0)*–*(A_2) *are satisfied. If there is a Dulac function of system* (1) *in the region* G*, then it holds* $\sharp\,\Gamma(G)\leqslant 1$.

The concept of the Dulac function has been generalized by L. Cherkas [3]. For this new class of functions we introduced in [7] the name Dulac–Cherkas function.

Definition 2. Suppose the assumptions (A_0) and (A_1) are satisfied. A function $\Psi\in C^1_{2\pi}(G,R)$ is called a Dulac–Cherkas function of system (1) in G if the set $W:=\{(x,y)\in G:\ \Psi(x,y)=0\}$ does not contain a curve which is a trajectory of system (1) and there is a real number $k\neq 0$ such that the following condition holds

$$\Phi(x,y,k):=(\operatorname{grad}\Psi,X)+k\Psi\operatorname{div}X\geqslant 0\ (\leqslant 0)\ \ \forall\,(x,y)\in G, \tag{2}$$

where the set $V_k:=\{(x,y)\in G:\ \Phi(x,y,k)=0\}$ has measure zero.

For $k=1$ the definition of a Dulac–Cherkas function coincides with the definition of a Dulac function. If Ψ is a Dulac–Cherkas function of system (1) in G, then $|\Psi|^{1/k}$ is a Dulac function of (1) in $G\setminus W$. For the following results we introduce the assumption.

(A_3). There is a Dulac–Cherkas function Ψ of system (1) in G with $k<0$ such that the set W consists of $l\geqslant 1$ simple closed curves $w_1,\dots,w_l$ surrounding the cylinder Z (we call them ovals) and which do not meet each other as well as the boundaries Δ_u and Δ_l of G.

Remark 1. If we consider the function Φ on any oval w_i of the set W, then we get from (2)

$$\Phi(x,y,k)_{|w_i}=(\operatorname{grad}\Psi,X)_{|w_i}=\frac{d\Psi}{dt}_{|w_i}\geqslant 0\ (\leqslant 0),$$

where d/dt denotes the differentiation along system (1). The conditions in Definition 2 implies

$$\frac{d\Psi}{dt}_{|w_i}\not\equiv 0,$$

and we can conclude that any trajectory of (1) which meets any oval w_i will cross it for increasing or decreasing t.

Concerning the location of these ovals on the cylinder Z we assume that the oval w_i is located over the oval w_{i+1}. The doubly connected subregion of G bounded by w_i and w_{i+1} is denoted by Z_i, $i = 1, \dots, l-1$, the region bounded by Δ_u and w_1 is denoted by Z_0, and the region bounded by w_l and Δ_l is denoted by Z_l, which are the outer regions.

The following result is also known [5].

Theorem 2. *Suppose that the assumptions* (A_0)*–*(A_3) *are valid. Then it holds:*

(i) *Each region* Z_i, $1 \leqslant i \leqslant l-1$, *contains a unique limit cycle* Γ_i *of the second kind of system* (1). Γ_i *is hyperbolic, it is stable (unstable) if* $\Phi(x,y,k)\Psi(x,y) > 0$ (< 0) *in* Z_i.

(ii) *The regions* Z_0 *and* Z_l *may contain a unique limit cycle of the second kind which is hyperbolic, and therefore, it implies immediately the estimate*

$$l - 1 \leqslant \sharp\,\Gamma(G) \leqslant l + 1. \tag{3}$$

Remark 2. Under the assumptions (A_0)–(A_3) any improvement of estimate (3) is connected with the existence or absence of a limit cycle of the second kind in the regions Z_0 and Z_l.

Now we want to establish conditions for the existence of a limit cycle of the second kind in Z_0 and/or in Z_l. By Remark 1 we can conclude that any trajectory of system (1) that meets an oval w_i of the set W will cross w_i for increasing or decreasing t. Therefore, appropriate Dulac–Cherkas functions can be used to construct doubly-connected regions to which the Poincaré–Bendixson theorem can be applied.

Theorem 3. *Suppose that the assumptions* (A_0)*–*(A_3) *are valid. Additionally, we assume the existence of a second Dulac–Cherkas function* Ψ_0 *of system* (1) *in some doubly connected subregion* $\widetilde{Z}_0$ *of* Z_0 *whose boundaries surround* Z *such that the corresponding set* $W_0 := \{(x,y) \in \widetilde{Z}_0 :\ \Psi_0(x,y) = 0\}$ *consists of exactly one oval* v_0 *and where the ovals* v_0 *and* w_1 *form the boundaries of the doubly connected region* Z_{00} *to which the Poincaré-Bendixson theorem can be applied. Then it holds*

$$l \leqslant \sharp\,\Gamma(G) \leqslant l + 1.$$

In the same way we can formulate the similar theorem for the region Z_l.

Remark 3. If the assumptions of Theorem 3 are fulfilled simultaneously for both regions Z_0 and Z_l, then it holds

$$\sharp\,\Gamma(G) = l + 1. \tag{4}$$

The exact number of limit cycles of the second kind in G can be also determined by means of an additional Dulac–Cherkas function defined in the same region G.

Theorem 4. *Suppose the assumptions* (A_0)*–*(A_3) *are valid. Additionally, we assume the existence of a second Dulac–Cherkas function* Ψ_1 *of system* (1) *in* G *with* $k_1 < 0$ *such that the corresponding set* W_1 *consists of* $l+2$ *ovals. Then estimate* (4) *holds.*

As the next step we present another approach based on factorized Dulac functions.

Let χ_1 and χ_2 be functions of the space $C^1_{2\pi}(G,R)$. For the following, we introduce the sets

$$U_i := \big\{(x,y) \in G :\ \chi_i(x,y) = 0\big\},\ \ i = 1, 2.$$

We denote by U the set $U := U_1 \cup U_2$ and define the function $D : G \setminus U \to R^+$ by

$$D(x,y,k_1,k_2) := |\chi_1(x,y)|^{k_1} |\chi_2(x,y)|^{k_2}, \tag{5}$$

where k_1 and k_2 are real numbers. For the divergence of the vector field we get from (5) in the region $G \setminus U$

$$\operatorname{div}(DX) = |\chi_1|^{k_1-1}|\chi_2|^{k_2-1} \operatorname{sgn}\chi_1 \operatorname{sgn}\chi_2 \big(\chi_1\chi_2 \operatorname{div} X + k_1\chi_2(\operatorname{grad}\chi_1, X) + k_2\chi_1(\operatorname{grad}\chi_2, X)\big).$$

Our goal is to derive conditions such that D is a Dulac function in some region of $G \backslash U$. Therefore, additionally we suppose

(C_1). There are functions $\chi_1, \chi_2 \in C^1_{2\pi}(G, R)$ and real numbers k_1, k_2 such that in G the following condition holds

$$\Theta(x, y, k_1, k_2) := \chi_1\chi_2 \operatorname{div} X + k_1\chi_2(\operatorname{grad}\chi_1, X) + k_2\chi_1(\operatorname{grad}\chi_2, X) < 0 \ (> 0).$$

Since we are interested in estimating the number of limit cycles of the second kind in G, we assume

(C_2). The set U consists in G of n ovals surrounding Z.

We denote by $v_1, \dots, v_m$ the ovals of U, where v_i is located above v_{i+1}. We denote by Z_i, $1 \leqslant i \leqslant n-1$, the open doubly connected region bounded by v_i and v_{i+1}, Z_0 is the open doubly connected region bounded by Δ_u and v_1, Z_n is the open doubly connected region bounded by v_n and Δ_l.

Theorem 5. *Suppose the assumptions* (A_0)*,* (A_1)*,* (A_2) *and* (C_1) *with* $k_1 < 0, k_2 < 0$*, and* (C_2) *are valid. Then it holds:*

(i) *Each region* Z_i*,* $1 \leqslant i \leqslant n-1$*, contains a unique limit cycle* Γ_i *of the second kind of system* (1)*.* Γ_i *is hyperbolic and stable (unstable) if the inequality*

$$\frac{\Theta(x, y, k_1, k_2)}{\chi_1(x, y)\chi_2(x, y)} < 0 \ (> 0)$$

is valid in Z_i*.*

(ii) *In each of the regions* Z_0 *and* Z_n *a unique hyperbolic limit cycle of the second kind could be located.*

A detailed presentation of our approaches to check the existence of a limit cycle in the regions Z_0 and Z_l or Z_n by means of an additional Dulac–Cherkas functions or by special factorized Dulac functions and their application to some classes of systems (1) are contained in our paper [6].

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Description by Suslin's Sets of Bounded Families of Liapunov's Characteristic Exponents in the Full Perron's Effect of Their Value Change

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We consider the linear differential system

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^2, \;\; t \geq 0, \tag{1}$$

with a bounded continuously differentiable matrix of coefficients $A(t)$ and with negative characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$. The system is a linear approximation for the nonlinear system

$$\dot{y} = A(t)y + f(t,y), \quad y = (y_1, y_2)^\top \in \mathbb{R}^2, \;\; t \geq 0. \tag{2}$$

In addition, the so-called m-perturbation $f(t,y)$ is continuously differentiable in its arguments $t \geq 0$ and $y_1, y_2 \in \mathbb{R}$ and has an $m \geq 1$ order of smallness in some neighbourhood of the origin and growth outside of it:

$$\|f(t,y)\| \leq C_f \|y\|^m, \;\; m > 1, \;\; y \in \mathbb{R}^2, \;\; t \geq 0. \tag{3}$$

Perron's effect [7], [6, pp. 50, 51] of sign and value change of characteristic exponents establishes the existence of system (1) with negative Lyapunov exponents and 2-perturbation (3) such that all nontrivial solutions of the perturbed system (2) turn out to be infinitely continuable and have finite Lyapunov exponents equal to:

(1) the negative higher exponent λ_2 of the initial system (1) for solutions starting at the initial moment on the axis $y_1 = 0$ (that allows one to consider Perron's effect not full);

(2) a certain positive value for all the rest solutions (calculated in [2, pp. 13–15]).

A number of works written by the author and jointly with Korovin contain various versions of the full Perron's effect when all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) are infinitely continuable (this is not the case in a general case) and have finite positive Lyapunov exponents under negative exponents of the system of linear approximation (1). These versions correspond to different types of the set $\lambda(A,f) \subset (0,+\infty)$ of Lyapunov's characteristic exponents of all nontrivial solutions of the perturbed system (2), to distribution of these solutions with respect to the exponents from the set $\lambda(A,f)$ and, finally, to an arbitrary order of systems (1) and (2).

In particular, it is stated in [3, 4] that the sets $\lambda(A,f)$ in this full Perron's effect are Suslin's ones [1, pp. 97, 98, 192]. For a complete description of (bounded) families $\lambda(A,f) \subset (0,+\infty)$ in that effect there arises an inverse question on the realization of an arbitrary bounded Suslin's set $S \subset (0, +\infty)$ by the family $\lambda(A, f)$ of characteristic exponents of a certain perturbed system (2), i.e., the question on the realization of the equality $\Lambda(A_s, f_s) \equiv S$ for the above-mentioned matrix $A_s(t)$ and vector-function $f_s(t, y)$.

The positive and stronger answer to the above question in classes of infinitely differentiable matrices $A(t)$ and vector-functions $f(t, y)$ in the corresponding spaces (that will be additionally supposed in the sequel) is contained in the present report.

The following theorem is valid.

Theorem 1 ([5]). *For arbitrary parameters $m > 1$, $\lambda_1 \le \lambda_2 < 0$ and arbitrary bounded on the axis $\mathbb{R}_0 = \mathbb{R} \setminus 0$ Baer's 1st class functions*

$$\psi_i : \mathbb{R}_0 \to [\beta_i, b_i] \subset (0, +\infty), \quad b_1 \le \beta_2, \quad i = 1, 2,$$

there exist a linear system (1) *with bounded infinitely differentiable on the semi-axis* $[t_0, +\infty)$ *coefficients and exponents $\lambda_1(A) = \lambda_1 \le \lambda_2 = \lambda_2(A)$ and the infinitely differentiable in its arguments $t \ge t_0$ and $y_1, y_2 \in \mathbb{R}$ m-perturbation $f(t, y)$ such that all nontrivial solutions $t(t, c)$ of the nonlinear system* (2) *are infinitely continuable to the right and have characteristic exponents*

$$\lambda[y(\,\cdot\,, c)] = \begin{cases} \psi_1(c_1), & c_1 \ne 0, \ \ c_2 = 0, \\ \psi_2(c_2), & c_2 \ne 0, \ ; \ c = (c_1, c_2) \in \mathbb{R}^2. \end{cases}$$

The above theorem results in the following corollary.

Corollary 1 ([5]). *For arbitrary parameters $m > 1$, $\lambda_1 \le \lambda_2 < 0$ and the bounded Suslin's set $S \subset (0, +\infty)$ there exist systems* (1) *and* (2) *mentioned in the above theorem such that the set of characteristic exponents of nontrivial solutions of the latter coincides with the set S.*

When proving the theorem we have used the following statements.

Lemma 1 ([5]). *Let the bounded on the axis $R_0 = R \setminus \{0\}$ function*

$$\psi : R_0 \to |\beta_0, b_0|, \quad -\infty < \beta_0 < b_0 < +\infty,$$

be Baer's 1st class function. Then for arbitrary constants $\beta < \beta_0$ and $b > b_0$ there exists a sequence $\{\psi_n(x)\}$ of infinitely differentiable uniformly bounded on the axis $\mathbb{R}_0$ functions $\psi_n : R_0 \Longrightarrow [\beta, b]$, $n \in \mathbb{N}$, converging on that axis to the function $\psi(x)$.

Lemma 2 ([5]). *For arbitrary numbers $\varepsilon > 0$ and continuous on the axis $\mathbb{R}_0$ function $F_0 : \mathbb{R}_0 \to \mathbb{R}$ there exists an infinitely differentiable on that axis function $F : \mathbb{R}_0 \to \mathbb{R}$ for which the inequality*

$$|F(x) - F_0(x)| \le \varepsilon, \quad x \in \mathbb{R}_0$$

is fulfilled.

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On Investigation and Approximate Solution of One System of Nonlinear Two-Dimensional Partial Differential Equations

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Systems of nonlinear partial differential equations are describing many real processes. The present note is devoted to one of such mathematical model arising in the investigation of the vein formation in leaves of higher plants and is represented as the two-dimensional nonlinear partial differential system [7]:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\Big(V\,\frac{\partial U}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(W\,\frac{\partial U}{\partial y}\Big),\\
\frac{\partial V}{\partial t} &= -V + G\Big(V\,\frac{\partial U}{\partial x}\Big),\\
\frac{\partial W}{\partial t} &= -W + H\Big(W\,\frac{\partial U}{\partial y}\Big),
\end{aligned} \tag{1}$$

where $U = U(x,y,t)$, $V = V(x,y,t)$, $W = W(x,y,t)$ are unknown functions defined on the domain $\overline{Q} = \overline{\Omega} \times [0,T] = [0,1] \times [0,1] \times [0,T]$, $T = Const > 0$ and G, H are known functions of their arguments.

In $\overline{Q}$ we consider initial-boundary value problems for (1) and for the following parabolic type regularization of system (1):

$$\begin{aligned}
\frac{\partial U_\varepsilon}{\partial t} &= \frac{\partial}{\partial x}\Big(V_\varepsilon\,\frac{\partial U_\varepsilon}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(W_\varepsilon\,\frac{\partial U_\varepsilon}{\partial y}\Big),\\
\frac{\partial V_\varepsilon}{\partial t} &= -V_\varepsilon + G\Big(V_\varepsilon\,\frac{\partial U_\varepsilon}{\partial x}\Big) + \varepsilon\,\frac{\partial^2 V_\varepsilon}{\partial x^2}\,,\\
\frac{\partial W_\varepsilon}{\partial t} &= -W_\varepsilon + H\Big(W_\varepsilon\,\frac{\partial U_\varepsilon}{\partial y}\Big) + \varepsilon\,\frac{\partial^2 W_\varepsilon}{\partial y^2}\,,
\end{aligned} \tag{2}$$

with the first type boundary conditions for U, U_ε and the second type boundary conditions for V_ε and W_ε. In (2) we assume that $\varepsilon = Const > 0$.

Some properties of the solutions of initial-boundary problems for systems (1) and (2) are studied.

The convergence of the solution of initial-boundary value problem of the regularized system (2) as $\varepsilon \to 0$ to corresponding solution of model (1) in the norm of the space $L_2(\Omega)$ is discussed.

For building approximate solutions of considered problems two different approaches are used. Both belong to the so-called decomposition methods [8]. The first approach is a decomposition method based on the variable directions difference scheme [1] and the second approach is based on averaged model [8]. The stability and convergence of these schemes are analyzed.

The one-dimensional (1) type system at first has been investigated in [2] and multi-dimensional one in [3,4]. For a brief overview of some research devoted to (1), (2), and relative models we refer to the papers [5,6].

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Representation of the Solution of the Inhomogeneous Wave Equation in a Half-Strip in the Form of Finite Sum of Addends, Depending on Boundary, Initial Values of the Solution and Right-Hand Side of the Equation

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In a plane of independent variables x and t in the half-strip $D_\infty:\ 0<x<l,\ t>0$ consider the mixed problem of finding solution $u(x,t)$ of the linear inhomogeneous wave equation of the form

$$\Box\, u=f(x,t),\quad (x,t)\in D_\infty, \tag{1}$$

satisfying the following initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\leq x\leq l, \tag{2}$$

and boundary conditions

$$u(0,t)=\mu_1(t),\ \ t\geq 0, \tag{3}$$

$$u(l,t)=\mu_2(t),\ \ t\geq 0, \tag{4}$$

where f, φ, ψ, μ_i, $i=1,2$, are given functions and u is unknown real function, and $\Box:=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$.

It is easy to see that for

$$f\in C^1(\overline{D}_\infty),\ \ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \mu_i\in C^2([0,\infty)),\ \ i=1,2,$$

the necessary conditions for solvability of problem (1)–(4) in the class $C^2(\overline{D}_\infty)$ are the following second order agreement conditions

$$\begin{gathered}\varphi(0)=\mu_1(0),\ \ \psi(0)=\mu_1'(0),\ \ \mu_1''(0)-\varphi''(0)=f(0,0),\\ \varphi(l)=\mu_2(0),\ \ \psi(l)=\mu_2'(0),\ \ \mu_2''(0)-\varphi''(l)=f(l,0).\end{gathered}$$

Let

$$m=m(t):=\Big[\frac{t}{l}\Big],\ \ t>0,$$

where $[\cdot]$ is an integer part of a real number.

Let us divide the domain $E_m:\ 0<x<l,\ ml<t<(m+1)l,\ m=0,1,2,\dots$, which is a quadrat with vertices in points $A_m(0,ml)$, $B_m(0,(m+1)l)$, $C_m(l,(m+1)l)$ and $D_m(l,ml)$ into four rectangular triangles: $E_m^1:=\Delta A_mO_mD_m$, $E_m^2:=\Delta A_mO_mB_m$, $E_m^3:=\Delta D_mO_mC_m$ and $E_m^4:=\Delta B_mO_mC_m$, where point $O_m(\frac{l}{2},(m+\frac{1}{2})l)$ is a center of the quadrat E_m.

Below we get the representation of the classical solution $u\in C^2(\overline{D}_\infty)$ of problem (1)–(4) in the half-strip D_∞ in the form of finite sum of addends, depending on boundary, initial values of this solution and right-hand side of equation (1).

First let $P = P(x,t) \in E_0$. In the triangle E_0^1 due to (2) and the d'Alembert's formula, the equality [7, p. 59]

$$u(x,t) = A_1(\varphi,\psi,f)(x,t) := \frac{1}{2}\big[\varphi(x-t)+\varphi(x+t)\big] + \frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^1} f(\xi,\tau)\,d\xi\,d\tau,\ \ (x,t)\in E_0^1 \tag{5}$$

is valid, where $\Omega_{x,t}^1$ is the triangle with vertices at the points (x,t), $(x-t,0)$ and $(x+t,0)$.

As it is known, for any twice continuously differentiable function v and characteristic to equation (1) rectangle $PP_1P_2P_3$ from its domain of definition the equality [1, p. 173]

$$v(P) = v(P_1) + v(P_2) - v(P_3) + \frac{1}{2}\int\limits_{PP_1P_2P_3} \Box\, v(\xi,\tau)\,d\xi\,d\tau \tag{6}$$

is valid, where P and P_3, P_1 and P_2 are opposite vertices of this rectangle, and the ordinate of point P is larger than those of the rest points.

Let now $P \in E_0^2$. Then, using equality (6) for characteristic rectangle with vertices at the points $P(x,t)$, $P_1(0,t-x)$, $P_2(t,x)$ and $P_3(t-x,0)$, and formula (5) for point $P_2(t,x) \in E_0^1$, in view of (1) and (3) we have

$$u(x,t) = A_2(\varphi,\psi,\mu_1,f)(x,t) := \mu_1(t-x) + \frac{1}{2}\big[\varphi(t+x)-\varphi(t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{t+x}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^2} f(\xi,\tau)\,d\xi\,d\tau,\ \ (x,t)\in E_0^2. \tag{7}$$

Here $\Omega_{x,t}^2$ is the quadrangle $PP_2^*P_3P_1$, where $P_2^* := (t+x,0)$.

Analogously, we have

$$u(x,t) = A_3(\varphi,\psi,\mu_2,f)(x,t) := \mu_2(x+t-l) + \frac{1}{2}\big[\varphi(x-t)-\varphi(2l-x-t)\big] + \frac{1}{2}\int\limits_{x-t}^{2l-x-t}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^3} f(\xi,\tau)\,d\xi\,d\tau,\ \ (x,t)\in E_0^3, \tag{8}$$

and

$$u(x,t) = A_4(\varphi,\psi,\mu_1,\mu_2,f)(x,t) := \mu_1(t-x) + \mu_2(x+t-l) - \frac{1}{2}\big[\varphi(t-x)+\varphi(2l-t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{2l-t-x}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^4} f(\xi,\tau)\,d\xi\,d\tau,\ \ (x,t)\in E_0^4. \tag{9}$$

Here $\Omega_{x,t}^3$ is a quadrangle with vertices $P^3(x,t)$, $P_1^3(l,x+t-l)$, $P_2^3(x-t,0)$ and $P_3^3(2l-x-t,0)$, while $\Omega_{x,t}^4$ is a pentagon with vertices $P^4(x,t)$, $P_1^4(0,t-x)$, $P_2^4(t-x,0)$, $P_3^4(2l-x-t,0)$ and $P_4^4(l,x+t-l)$.

If the point $P_0 := P_0(x,t) \in E_m$, $m \geq 1$, then denote by $P_0M_1P_1N_1$ the characteristic rectangle with respect to equation (1), whose vertices M_1 and N_1 lay on the straight lines $x=0$ and $x=l$, respectively, i.e. $M_1 := (0,t-x)$, $N_1 := (l,t+x-l)$, $P_1 := (l-x,t-l)$. Since $P_1 \in E_{m-1}$, then by analogy we can consider the characteristic rectangle $P_1M_2P_2N_2$, whose vertices M_2 and N_2 lay on the straight lines $x=0$ and $x=l$, respectively. Continuing this process we get the characteristic rectangle $P_{i-1}M_iP_iN_i$ with vertices M_i and N_i, respectively, on the straight lines $x=0$ and $x=l$, and due to $P_0\in E_m$,

$$P_m\in E_0, \tag{10}$$

where $P_m=(l-x,t-ml)$ if m is odd, and $P_m=(x,t-ml)$ if m is even. At the same time if the point $P_0\in E^1_m(E^4_m)$, then $P_m\in E^1_0(E^4_0)$ for any m, and if $P_0\in E^2_m(E^3_m)$, then $P_m\in E^3_0(E^2_0)$ for odd m and $P_m\in E^2_0(E^3_0)$ for even m. For the coordinates of the points M_i and N_i we have

$$\begin{gathered}M_i=(0,t-x-(i-1)l),\ \ N_i=(l,t+x-il),\ \ i=1,3,5,\dots,\\ M_i=(0,t+x-il),\ \ N_i=(l,t-x-(i-1)l),\ \ i=2,4,6,\dots.\end{gathered}$$

By induction over number m it can be proved the validity of the following representation of the solution $u\in C^2(\overline{D}_\infty)$ of problem (1)–(4) in the half-strip D_∞

$$u(P_0)=\sum_{i=1}^m(-1)^{i-1}\bigg[\mu_1(M_i)+\mu_2(N_i)+\frac12\int\limits_{P_{i-1}M_iP_iN_i}f(\xi,\tau)\,d\xi\,d\tau\bigg]+(-1)^mu(P_m),\ \ P_0\in E_m,$$

where due to (10) in the case of odd m

$$u(P_m)=\begin{cases}A_1(\varphi,\psi,f)(P_m), & P_0\in E^1_m,\\ A_3(\varphi,\psi,\mu_2,f)(P_m), & P_0\in E^2_m,\\ A_2(\varphi,\psi,\mu_1,f)(P_m), & P_0\in E^3_m,\\ A_4(\varphi,\psi,\mu_1,\mu_2,f)(P_m), & P_0\in E^4_m,\end{cases}$$

while for even m

$$u(P_m)=\begin{cases}A_1(\varphi,\psi,f)(P_m), & P_0\in E^1_m,\\ A_2(\varphi,\psi,\mu_1,f)(P_m), & P_0\in E^2_m,\\ A_3(\varphi,\psi,\mu_2,f)(P_m), & P_0\in E^3_m,\\ A_4(\varphi,\psi,\mu_1,\mu_2,f)(P_m), & P_0\in E^4_m.\end{cases}$$

Here the operators A_i, $i=1,2,3,4$ are defined by formulas (5), (7)–(9).

The obtained representation will unconditionally find application during a study of other initial-boundary value problems both for linear and nonlinear hyperbolic equations and systems. Let us note that other representations of the solution of problem (1)–(4) in the form of infinite series are given in [1–9].

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Regularization Method in Stability Analysis of Stochastic Functional Differential Equations

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Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space. By $Z:=(z_1,\dots,z_m)^T$ we denote an m-dimensional semimartingale (see, e.g. [7]). A popular example of such Z is the vector Brownian motion (the Wiener process). The linear space k^n consists of all n-dimensional $\mathcal{F}_0$-measurable random variables.

The main idea of the method, which is outlined below, is to represent the property of Lyapunov stability in terms of invertibility of certain linear operators in suitable functional spaces.

The following linear homogeneous stochastic delay differential equation is considered

$$dx(t)=(V_hx)(t)\,dZ(t)\ \ (t\geq 0) \tag{1}$$

endowed with two initial conditions

$$x(s)=\varphi(s)\ \ (s<0) \tag{2}$$

and

$$x(0)=b. \tag{3}$$

Here V_h is a k-linear Volterra operator which is defined in certain linear spaces of vector stochastic processes, φ is an $\mathcal{B}(-\infty,0)\otimes\mathcal{F}_0$-measurable stochastic process and $b\in k^n$. By k-linearity of the operator V_h we mean the following property

$$V_h(\alpha_1x_1+\alpha_2x_2)=\alpha_1V_hx_1+\alpha_2V_hx_2,$$

which holds for all $\mathcal{F}_0$-measurable, bounded and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V_h. The exact assumptions on the domain and the range of V_h are specified below in connection with the properties of the associated operator V.

The solution of the initial value problem (1)–(3) will be denoted by $x(t,b,\varphi)$, $t\in(-\infty,\infty)$. The solution is always assumed to exist and to be unique for an appropriate choice of $\varphi(s),b$: for specific conditions see e.g. [3].

According to the habilitation thesis [3], the following classes of linear stochastic equations are particular cases of Eq. (1):

(A) *Systems of linear ordinary (i.e. non-delay) stochastic differential equations driven by an arbitrary semimartingale (in particular, systems of ordinary Itô equations)*;

(B) *Systems of linear stochastic differential equations with discrete delays driven by a semimartingale (in particular, systems of Itô equations with discrete delays)*;

(C) *Systems of linear stochastic differential equations with distributed delays driven by a semimartingale (in particular, systems of Itô equations with distributed delays)*;

(D) *Systems of linear stochastic integro-differential equations driven by a semimartingale (in particular, systems of Itô integro-differential equations)*;

(E) *Systems of linear stochastic functional difference equations driven by a semimartingale (in particular, systems of Itô functional difference equations).*

Definition 1. For a given real number q $(1 \le q < \infty)$ we call the zero solution of Eq. (1)

- q-stable (with respect to the initial data b and φ) if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $E|b|^q + \operatorname*{ess\,sup}\limits_{s<0} E|\varphi(s)|^q < \delta$ implies $E|x(t,b,\varphi)|^q \le \varepsilon$ for all $t \ge 0$ and all $\mathcal{F}_0$-measurable φ, b;
- exponentially q-stable if there exist positive constants K, λ such that the inequality

$$E|x(t,b,\varphi)|^q \le K\big(E|b|^q + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q\big)\exp\{-\lambda s\}$$

  holds true for all $t \ge 0$ and all $\mathcal{F}_0$-measurable φ, b.

Let S^n be a linear subspace of the space of $\mathcal{F}_t$-adapted, n-dimensional stochastic processes whose trajectories belong to a normed space E with the norm $\|\cdot\|_E$. Then we denote by S^n_q $(1 \le q < \infty)$, the linear subspace of S^n containing all processes $f \in S^n$, for which the norm defined by $\|f\|^q_{S^n_q} = E\|f\|^q_E$ is finite.

For instance, if Φ^n stands for all $\mathcal{F}_0$-measurable, n-dimensional prehistory functions φ with essentially bounded trajectories, then the norm in Φ^n_q is given by

$$\|\varphi\|_q = \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q.$$

This simplifies the notation in Definition 1, where the expression $E|b|^q + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^q$ may be replaced by $\|b\|^q_{k^n_q} + \|\varphi\|^q_{\Phi^n_q}$.

To describe the regularization method, one needs to represent (1)–(2) in a canonical form [1,3]. Let $x(t)$ be a stochastic process on $[0,+\infty)$ and $x_+(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $x(t)$ for $t \ge 0$ and equalling 0 for $t < 0$, while $\varphi_-(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $\varphi(t)$ for $t < 0$ and equalling 0 for $t \ge 0$. Then the stochastic process $x_+(t) + \varphi_-(t)$, defined for $t \in (-\infty,+\infty)$ will be a solution of the problem (1)–(3) if $x(t)$ $(t \in [0,+\infty))$ satisfies the initial value problem

$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t) \ \ (t \ge 0), \tag{4}$$

$$x(0) = b, \tag{5}$$

where $(Vx)(t) := (V_h x_+)(t)$, $f(t) := (V_h\varphi_-)(t)$ for $t \ge 0$. Indeed, by k-linearity we have that $V_h(x_+ + \varphi_-) = V_h(x_+) + V_h(\varphi_-) = Vx + f$, which gives (4). Note that f is uniquely defined by the prehistory function φ. Let us also observe that the initial value problem (4)–(5) is equivalent to the initial value problem (1)–(3) only for f, which have the representation $f = V_h\varphi'$, where φ' is an arbitrary extension of the function φ to the real line $(-\infty,\infty)$.

The solution of (4)–(5) is below denoted by $x_f(t,b)$.

Let B^n be a linear subspace of the space of $\mathcal{F}_t$-adapted stochastic processes with trajectories which are almost surely essentially bounded on $[0,\infty)$. According to our notation, the norm in the space B^n_q is defined by

$$\|f\|^q_{B^n_q} = \operatorname*{ess\,sup}_{t\geq 0} E|f(t)|^q.$$

Let $L^n(Z)$ be the set of all $n\times m$-matrix predictable stochastic processes defined on $[0,+\infty)$ and whose rows are locally integrable with respect to the semimartingale Z, see e.g. [3], and D^n be the set of all n-dimensional stochastic processes on $[0,+\infty)$, which can be represented as

$$x(t) = x(0) + \int\limits_0^t H(s)\,dZ(s),$$

where $x(0)\in k^n$, $H\in L^n(Z)$. The space D^n and its linear subspaces D^n_q are called the spaces of solutions of Eq. (4) (see [3]). The operator V is usually assumed to be a bounded linear operator from D^n_q to $L^n_q(Z)$ for some $1\leq q<\infty$.

This yields two linear operators

$$\mathcal{L}_1:\varphi\longmapsto (V_h\varphi_-)(t) \tag{6}$$

and

$$\mathcal{L}_2: f\longmapsto x_f(\,\cdot\,,b). \tag{7}$$

The following result is crucial for the framework (see e.g. [5]).

Theorem 1. *Assume that the linear operators $\mathcal{L}_1:\Phi_q\to B^n_q$ and $\mathcal{L}_2: B^n_q\to D^n_q$ are bounded. Then the zero solution of Eq.* (1) *is q-stable in the sense of Definition 1.*

In applications, the operator $\mathcal{L}_1$ is usually bounded, so that the only challenge in application of Theorem 1 is to prove boundedness of the operator $\mathcal{L}_2$. This can be done by the regularization method called in [1] and [3] 'the W-method'. The regularization is usually constructed with the help of an auxiliary equation

$$dx(t) = \big[(Qx)(t)+g(t)\big]\,dZ(t)\ \ (t\geq 0), \tag{8}$$

where Q is again a k-linear Volterra operator. This equation is similar to Eq. (4), possesses the existence and uniqueness property as well, but it is usually chosen to be 'simpler' in the sense that the required stability property for this equation is already known (see assumption (2) in Theorem 2 below).

The following representation formula for the solutions of Eq. (8) can be directly deduced from the existence and uniqueness property

$$x(t) = U(t)x(0) + (Wg)(t)\ \ (t\geq 0), \tag{9}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator.

Using representation (9) we can regularize Eq. (4). This algorithm is based on the framework described in [3,5].

Using Eq. (8) we rewrite Eq. (4) as follows

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]\,dZ(t)\ \ (t\geq 0),$$

or, taking (9) into account, as

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t) \ \ (t \geq 0).$$

Putting $W(V-Q) = \Theta$, we obtain the operator equation

$$x(t) = (\Theta x)(t) + U(t)x(0) + (Wf)(t) \ \ (t \geq 0). \tag{10}$$

Theorem 2. *Assume that Eq.* (4) *and the reference equation* (8) *satisfy the following conditions:*

(1) *the linear operators V, Q act continuously from D_q^n to B_q^n;*

(2) *the Cauchy operator W in* (9) *constructed for the reference equation* (8) *is bounded as an operator from B_q^n to D_q^n;*

(3) *the operator $I - \Theta : D_q^n \to D_q^n$ has a bounded inverse.*

Then the operator $\mathcal{L}_2 : B_q^n \to D_q^n$ in (7) *is bounded.*

Theorems 1 and 2 justify the regularization method for Lyapunov stability of stochastic linear functional differential equations. The main challenge of the method is to prove that the operator $I - \Theta$ has a bounded inverse. In [3–5] (see also the references therein) this property is checked by estimating the norm of the integral operator Θ: if $\|\Theta\|_{D_q^n} < 1$ in the inequality

$$\|x\|_{D_q^n} \leq \|\Theta\|_{D_q^n} \|x\|_{D_q^n} + K_1 \|x(0)\|_{k_q^n} + K_2 \|f\|_{B_q^n}, \tag{11}$$

then Eq. (1) is q-stable due to Theorem 1. Moreover, if $q \geq 2$ and the equation remains q-stable after the substitution $y(t) = \exp(\lambda t)x(t)$ for some positive λ, then Eq. (1) is, in fact, exponentially q-stable.

Another approach, which has recently been suggested in [2] in the deterministic case and in [6] in the stochastic case, is based on the properties of monotone operators. In this case, the estimation is done componentwise, and if the resulting matrix has a bounded inverse, then one still obtains inequalities like (11). A short description of this method is given below.

Recall that an $m \times m$-matrix $B = (b_{ij})_{i,j=1}^m$ is said to be nonnegative, resp. positive if $b_{ij} \geq 0$, resp. $b_{ij} > 0$ for all $i, j = 1, \dots, m$.

Definition 2. A matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called a (non-singular) $\mathcal{M}$-matrix if $\gamma_{ij} \leq 0$ for $i, j = 1, \dots, n$, $i \neq j$, and all the principal minors of the matrix Γ are positive.

Let

$$x(t) = \operatorname{col}(x_1(t), \dots, x_n(t)), \ \ \overline{x}_i = \sup_{t \geq 0} \big(E|x_i(t)|^q \big)^{1/q}, \ \ \overline{x} = \operatorname{col}(\overline{x}_1, \dots, \overline{x}_n).$$

Suppose that after componentwise estimation in the vector equation (10) we get the following vector inequality

$$D\overline{x} \leq \|x(0)\|_{k_q^n} \overline{e}_1 + \|f\|_{B_q^n} \overline{e}_2, \tag{12}$$

where D is an $n \times n$-matrix, $\overline{e}_1$, $\overline{e}_2$ are some column n-vectors with nonnegative components. Typically, $D = \overline{E} - T$, where $\overline{E}$ is the $n \times n$ identity matrix, while T and $\overline{e}_i$ replace Θ and K_i $(i = 1, 2)$ in the scalar inequality (11), respectively. Then we obtain

Theorem 3. *If D is an $\mathcal{M}$-matrix in the sense of Definition 2, then the operator $\mathcal{L}_2 : B_q^n \to D_q^n$ in* (7) *is bounded.*

Proof. As D is an $\mathcal{M}$-matrix, the matrix D^{-1} is positive, and we can rewrite (12) as

$$\overline{x} \leq D^{-1}\big(\|x(0)\|_{k_q^n}\overline{e}_1 + \|f\|_{B_q^n}\overline{e}_2\big).$$

Therefore,

$$|\overline{x}| \leq K\big(\|x(0)\|_{k_q^n} + \|f\|_{B_q^n}\big), \tag{13}$$

where $K = \|D^{-1}\| \max\{|e_1|, |e_2|\}$. As $\|x\|_{D_q^n} \leq |\overline{x}|$, we conclude from (13) that $x \in D_q^n$ and $\|x\|_{D_q^n} \leq K(\|b\|_{k_q^n} + \|f\|_{B_q^n})$ for some positive K. Thus, the operator $\mathcal{L}_2 :\ B_q^n \to D_q^n$ is bounded. $\square$

Again, if $q \geq 2$ and one uses the substitution $y(t) = \exp(\lambda t)x(t)$ for some positive λ and Theorems 1, 3 and proves q-stability of the equation for $y(t)$, then this result will imply exponential q-stability of Eq. (1).

The outlined frameworks can be applied to all systems of stochastic differential equations mentioned above as classes (A)–(E). Notice that the second Lyapunov method might be difficult to use in many of these cases.

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Solvability of the Boundary Value Problem for One Class of Higher-Order Nonlinear Partial Differential Equations

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In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,x_2,\dots,x_n)$ and t we consider the nonlinear equation of the type

$$L_f u := \frac{\partial^{2(2k+1)}u}{\partial t^{2(2k+1)}} - \Delta^2 u + f(u,\nabla u) = F(x,t), \tag{1}$$

where f and F are given, and u is an unknown real functions, $\nabla := (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t})$, $\Delta := \sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, k is a natural number and $n\geq 2$.

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, a solution $u=u(x,t)$ of that equation according to the boundary conditions

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0, \ \ i=0,\dots,2k, \tag{2}$$

$$u\big|_{\Gamma_T} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma_T} = 0, \tag{3}$$

where $\Gamma_T := \partial\Omega\times(0,T)$ is the lateral face of the cylinder D_T, $\Omega_0:\ x\in\Omega,\ t=0$ and $\Omega_T:\ x\in\Omega,\ t=T$ are bottom and top bases of this cylinder, respectively, and $\frac{\partial}{\partial\nu}$ is a derivative along the outer normal to the boundary ∂D_T of the domain D_T. For $T=\infty$ we have $D_\infty=\Omega\times(0,\infty)$, $\Gamma_\infty=\partial\Omega\times(0,\infty)$.

Note that the linear part of the operator L_f from (1), i.e. L_0 is a hypoelliptic operator.

Below, for function $f=f(s_0,s_1,\dots,s_{n+1}),(s_0,s_1,\dots,s_{n+1})\in\mathbb{R}^{n+2}$ we assume that

$$f\in C(\mathbb{R}^{n+2}) \tag{4}$$

and

$$\big|f(s_0,s_1,\dots,s_{n+1})\big| \leq M + \sum_{i=0}^{n+1} M_i|s_i|^{\alpha_i} \ \ \forall\, s=(s_0,s_1,\dots,s_{n+1})\in\mathbb{R}^{n+2}, \tag{5}$$

where $M,M_i,\alpha_i = const > 0$, $i=0,1,\dots,n+1$.

Denote by $C^{4,4k+2}(\overline{D}_T)$ the space of continuous functions in $\overline{D}_T$ having continuous partial derivatives $\partial_x^\beta u$, $\frac{\partial^l u}{\partial t^l}$ in $\overline{D}_T$, where $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}$, $\beta=(\beta_1,\dots,\beta_n)$, $|\beta| = \sum\limits_{i=1}^{n}\beta_i \leq 4$; $l=1,\dots,4k+2$.

Assume

$$C_0^{4,4k+2}(\overline{D}_T,\partial D_T) := \Big\{u\in C^{4,4k+2}(\overline{D}_T):\ u\big|_{\Gamma_T} = \frac{\partial u}{\partial\nu}\Big|_{\Gamma_T} = 0,\ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0,\ i=0,\dots,2k\Big\}.$$

Let $u \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ be a classical solution of the problem $(1),(2),(3)$. Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$-\int\limits_{D_T} \Big[\frac{\partial^{2k+1} u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta \varphi\Big]\,dx\,dt + \int\limits_{D_T} f(u, \nabla u)\varphi\,dx\,dt \\ = \int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in C^{4,4k+2}(\overline{D}_T, \partial D_T). \tag{6}$$

We take the equality (6) as a basis for our definition of the weak generalized solution u of the problem $(1),(2),(3)$.

Introduce the Hilbert space $W_0^{2,2k+1}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{2,2k+1}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^{2k+1} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2\bigg]\,dx\,dt \tag{7}$$

of the classical space $C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$.

Remark 1. From (7) it follows that if $u \in W_0^{2,2k+1}(D_T)$, then $u \in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^l u}{\partial t^l} \in L_2(D_T)$; $i,j = 1,\dots,n$; $l = 1,\dots,2k+1$. Here $W_2^m(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having generalized derivatives from $L_2(D_T)$ up to m-th order inclusively, and $\overset{\circ}{W}{}_2^1(D_T) = \{u \in W_2^1(D_T) :\ u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory. Moreover, when the domain Ω is convex, and therefore the domain D_T is also convex, and since the following estimate

$$\int\limits_{D_T} \bigg[\sum_{i,j=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial^2 u}{\partial x_i \partial t}\Big)^2 + \Big(\frac{\partial^2 u}{\partial t^2}\Big)^2\bigg]\,dx\,dt \\ \leq c \int\limits_{D_T} \Big[\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2}\Big]^2\,dx\,dt \ \ \forall\,u \in \overset{\circ}{C}2(\overline{D}_T, \partial D_T) := \Big\{u \in C^2(\overline{D}_T) :\ u\big|_{\partial D_T} = 0\Big\}$$

holds with a positive constant c not dependant on u and the domain D_T, then from (7) we have continuous embedding of spaces

$$W_0^{2,2k+1}(D_T) \subset W_2^2(D_T). \tag{8}$$

Below, we assume that Ω is a convex domain.

Remark 2. As it is known the space $W_2^2(D_T)$ is continuously and compactly embedded into $L_p(D_T)$ for $p < \frac{2(n+1)}{n-3}$ when $n > 3$ and for any $p \geq 1$ when $n = 2,3$; analogously, the space $W_2^1(D_T)$ is continuously and compactly embedded into $L_q(D_T)$ if $q < \frac{2(n+1)}{n-1}$. Therefore, taking into account continuous embedding of the spaces (8), the inequality (5) and the properties of the Nemytski operators N_i, $i = 0,1,\dots,n+1$, acting by formula $N_i v = |v|^{\alpha_i}$, we get that the nonlinear operator $N : W_0^{2,2k+1}(D_T) \to L_2(D_T)$ acting by formula $Nu = f(u, \Delta u)$, will be continuous and compact if the nonlinearity exponent α_i in the right-hand side of the inequality (5) satisfies the following inequalities:

$$1 < \alpha_0 < \frac{n+1}{n-3} \ \text{ for } \ n > 3; \quad \alpha_0 > 1 \ \text{ for } \ n = 2, 3; \tag{9}$$

$$1 < \alpha_i < \frac{n+1}{n-1}\,, \ \ i = 1, \dots, n+1, \ \ n \geq 2. \tag{10}$$

Besides, from the above-mentioned remarks it follows that if $u \in W_0^{2,2k+1}(D_T)$, then $f(u, \nabla u) \in L_2(D_T)$ and for $u_m \to u$ in the space $W_0^{2,2k+1}(D_T)$ we have $f(u_m, \nabla u_m) \to f(u, \nabla u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_2(D_T)$. The function $u \in W_0^{2,2k+1}(D_T)$ is said to be a weak generalized solution of the problem (1), (2), (3) if for any $\varphi \in W_0^{2,2k+1}(D_T)$ the integral equality (6) is valid.

Notice that when the conditions (4), (5), (9) and (10) are fulfilled, if $u \in W_0^{2,2k+1}(D_T)$ and $\varphi \in W_0^{2,2k+1}(D_T)$, then according to Remark 2 we have $f(u, \nabla u) \in L_2(D_T)$, $\varphi \in L_2(D_T)$ and the second addend

$$\int\limits_{D_T} f(u, \nabla u)\varphi \, dx\, dt$$

in the left-hand side of the equality (6) is defined correctly.

It is not difficult to verify that if the solution of the problem (1), (2), (3) in the sense of Definition 1 belongs to the class $C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$, then it will also be a classical solution of this problem.

Definition 2. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$. We say that the problem (1), (2), (3) is globally solvable in the class $W_0^{2,2k+1}$ if for any $T > 0$ this problem has at least one weak generalized solution $u \in W_0^{2,2k+1}(D_T)$ in the sense of Definition 1.

Definition 3. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$. We say that the problem (1), (2), (3) is locally solvable in the class $W_0^{2,2k+1}$ if there exists a number $T_0 = T_0(F)$ such that for any positive $T < T_0$ this problem has at least one weak generalized solution $u \in W_0^{2,2k+1}(D_T)$ in the sense of Definition 1.

It is proved that when the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$ are fulfilled, then the problem (1), (2), (3) is locally solvable in the class $W_0^{2,2k+1}$ in the sense of Definition 3, and for some additional conditions on the problem's data, in certain cases the problem (1), (2), (3) is locally solvable whereas it is not globally solvable, and in other cases we have a global solvability in the sense of Definition 2.

The case of uniqueness of the solution of this problem in D_∞ is also considered.

Emden–Fowler Type Differential Equations Possessing Kurzweil's Property

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M. Jasný and J. Kurzweil [1,2] was the first who revealed the fact that unlike the second order linear differential equations, the Emden-Fowler type nonlinear differential equation

$$u'' = p(t)|u|^{\lambda}\operatorname{sgn}(u),$$

where $\lambda = const > 1$, and $p : [a,+\infty) \to]-\infty, 0[$ is a continuous function, may have simultaneously oscillatory and nonoscillatory solutions.

According to F. V. Atkinson's theorem [3], from the proven by J. Kurzweil [2] oscillation theorem it follows that if the function $t \mapsto t^{\frac{\lambda+3}{2}}|p(t)|$ is nondecreasing and

$$\int\limits_a^{+\infty} t|p(t)|\,dt < +\infty,$$

then the above-mentioned Emden–Fowler type equation along with oscillatory solutions has also separated from zero slowly growing solutions. Such type of theorems for different classes of superlinear and sublinear differential equations of second and fourth order have been proven in [4–8].

We have established unimprovable in a certain sense conditions guaranteeing the fact that the higher order Emden–Fowler type differential equation

$$u^{(n)} = p(t)|u|^{\lambda(|u|)}\operatorname{sgn}(u) \tag{1}$$

has Kurzweil's property. Here, $n > 3$, $p : [a,+\infty[\to \mathbb{R}$ is a function, Lebesgue integrable on every finite interval contained in $[a,+\infty[$, $a > 0$, and $\lambda : [0,+\infty[\to \mathbb{R}$ is a continuous function. Moreover, the function p satisfies the inequality

$$(-1)^{n-n_0}p(t) \le 0 \ \text{ for } \ t \ge a, \tag{2}$$

where n_0 is the integer part of number $\frac{n}{2}$, and the function λ satisfies either the condition

$$1 < \lambda(x) \le \lambda(y) \ \text{ for } \ 0 < x < y < +\infty, \tag{3}$$

or the conditions

$$\begin{gathered}\lambda(0) > 1, \ \ \lambda(x) \ge \lambda(y) \ \text{ for } \ 0 \le x < y < +\infty, \ \ -\infty < \lambda_0 = \lim_{x\to+\infty}\lambda(x) < 1,\\ \limsup_{x\to+\infty}(\lambda(x) - \lambda_0)\ln(x) < +\infty.\end{gathered} \tag{4}$$

Let $t_0 \in [a,+\infty[$. The solution $u : [t_0,+\infty[\to \mathbb{R}$ of equation (1) is said to be **proper** if it is not identically equal to zero in non of the neighborhood of $+\infty$.

The proper solution $u : [t_0,+\infty[\to \mathbb{R}$ is called:

1) **oscillatory** if it changes its sign in any neighborhood of $+\infty$ and **nonoscillatory**, otherwise;
2) **Kneser solution** if
$$u(t) \neq 0, \ \ (-1)^i u^{(i)}(t)u(t) \geq 0 \ \text{ for } \ t \geq t_0 \ \ (i = 1, \dots, n-1);$$
3) **vanishing at infinity** if the equality
$$\lim_{t \to +\infty} u(t) = 0$$
is fulfilled, and **separated from zero** if the inequality
$$\liminf_{t \to +\infty} |u(t)| > 0$$
is fulfilled;
4) **slowly growing** if
$$\limsup_{t \to +\infty} |u^{(n-1)}(t)| < +\infty$$
and **rapidly growing** if
$$\lim_{t \to +\infty} |u^{(n-1)}(t)| = +\infty.$$

Definition 1. Equation (1) **has property** K if it has a continuum of proper oscillatory solutions and a continuum of separated from zero slowly growing solutions.

Definition 2. Equation (1) **has property** K_0 if it has a continuum of proper oscillatory solutions, a continuum of separated from zero slowly growing solutions and a continuum of vanishing at infinity Kneser solutions.

Theorem 1. *Let n_0 be odd and along with* (2) *and* (3), *the condition*
$$\int_a^{+\infty} t^{n-2+\lambda(tx)} |p(t)|\, dt = +\infty \ \ \textit{for} \ \ x > 0 \tag{5}$$
be fulfilled. Then equation (1) *has property K if and only if*
$$\int_a^{+\infty} t^{n-1} |p(t)|\, dt < +\infty. \tag{6}$$

Theorem 1′. *Let $n = 2n_0 + 1$ ($n = 2n_0$), n_0 be odd and conditions* (2), (3), (5) *and* (6) *be fulfilled. Then every nonoscillatory proper solution of equation* (1) *is separated from zero Kneser solution (either is separated from zero Kneser solution, or rapidly growing solution).*

Theorem 2. *Let n_0 be even (odd) and along with* (2) *and* (4) *the condition*
$$\int_a^{+\infty} t^{n-n_0+(n_0-1)\lambda_0} |p(t)|\, dt = +\infty \tag{7}$$
be fulfilled. Then equation (1) *has property K (property K_0) if and only if*
$$\int_a^{+\infty} t^{(n-1)\lambda_0} |p(t)|\, dt < +\infty. \tag{8}$$

Theorem 2′. *Let n_0 be even (odd) and conditions* (2), (4), (7) *and* (8) *be fulfilled. Then every proper nonoscillatory solution of equation* (1) *is separated from zero slowly growing (either is separated from zero slowly growing, or vanishing at infinity Kneser solution).*

Example. Let

$$\lambda(x)=\lambda_0+\frac{\lambda_1}{1+|x|}\,,\quad \text{where}\ \ \lambda_0\in\,]-\infty,1[\,,\ \ \lambda_1>1-\lambda_0. \tag{9}$$

Then conditions (4) are fulfilled. Therefore, if n_0 is even (is odd) and the function p satisfies conditions (2), (7) and (8), then equation (1) has property K (property K_0). Moreover, every proper nonoscillatory solution of that equation is separated from zero slowly growing (either is separated from zero slowly growing, or vanishing at infinity Kneser solution).

Remark. Condition (7) in Theorems 2 and 2′ is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_0^{+\infty} t^{n-n_0+(n_0-1)\lambda_0+\varepsilon}|p(t)|\,dt=+\infty,$$

no matter how small $\varepsilon>0$ is.

Finally, it should be noted that in the case $n=3$ the question on the validity of Theorems 1 and 2 remains open. In particular, the following problems remain unsolved.

Problem 1. *Let $n=3$, $\lambda(x)\equiv\lambda_0>1$,*

$$p(t)\le 0\ \ \textit{for}\ \ t\ge a,\qquad \int\limits_a^{+\infty} t^{1+\lambda_0}|p(t)|\,dt=+\infty,\qquad \int\limits_a^{+\infty} t^2|p(t)|\,dt<+\infty.$$

Then, does equation (1) *have at least one proper oscillatory solution or not?*

Problem 2. *Let $n=3$ and along with* (9) *the conditions*

$$p(t)\le 0\ \ \textit{for}\ \ t\ge a,\qquad \int\limits_a^{+\infty} t^2|p(t)|\,dt=+\infty,\qquad \int\limits_a^{+\infty} t^{2\lambda_0}|p(t)|\,dt<+\infty$$

be fulfilled. Then, does equation (1) *have at least one proper oscillatory solution or not?*

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Dirichlet type Problem in a Smooth Convex Domain for Quasilinear Hyperbolic Equations of Fourth Order

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Let $\Omega=(0,\omega_1)\times(0,\omega_2)$ be an open rectangle, and let $\mathbf{D}$ be an *orthogonally convex* open domain with C^2 boundary inscribed in Ω such that

$$\begin{aligned}\mathbf{D}&=\{(x_1,x_2)\in\Omega:\ x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\}\\&=\big\{(x_1,x_2)\in\Omega:\ x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\},\end{aligned}$$

where $\gamma_i\in C([0,\omega_1])\cap C^2((0,\omega_1))$, $\eta_i\in C([0,\omega_2])\cap C^2((0,\omega_2))$ $(i=1,2)$, and

$$\gamma_1(\xi_1^*)=0,\quad \gamma_2(\xi_2^*)=\omega_2,\quad \eta_1(\zeta_1^*)=0,\quad \eta_2(\zeta_2^*)=\omega_1$$

for some $\xi_1^*,\xi_2^*\in[0,\omega_1]$ and $\zeta_1^*,\zeta_2^*\in[0,\omega_2]$.

In the domain $\mathbf{D}$ consider the problem

$$u^{(2,2)}=p_1(x_1,x_2)u^{(2,0)}+p_2(x_1,x_2)u^{(0,2)}+\sum_{j=0}^{1}\sum_{k=0}^{1}p_{jk}(x_1,x_2)u^{(j,k)}+q(x_1,x_2), \tag{1}$$

$$u(\eta_i(x_2),x_2)=\varphi_i(x_2)\ \ (i=1,2);\quad u^{(2,0)}(x_1,\gamma_i(x_1))=\psi_i''(x_1)\ \ (i=1,2), \tag{2}$$

where

$$u^{(j,k)}(x_1,x_2)=\frac{\partial^{j+k}u}{\partial x_1^j\partial x_2^k}\,,$$

$p_i\in C(\overline{\mathbf{D}})$ $(i=1,2)$, $p_{jk}\in C(\overline{\mathbf{D}})$ $(j,k=0,1)$, $q\in C(\overline{\mathbf{D}})$, $\phi_i\in C^2([0,\omega_2])$, $\psi_i\in C^2([0,\omega_1])$ $(i=1,2)$, $C^{m,n}(\overline{\mathbf{D}})$ is the Banach space of functions $u:\overline{\mathbf{D}}\to\mathbb{R}$, having continuous partial derivatives $u^{(i,j)}$ $(i=0,\dots,m;\ j=0,\dots,n)$, with the norm

$$\|u\|_{C^{m,n}(\overline{\mathbf{D}})}=\sum_{j=0}^{m}\sum_{k=0}^{n}\|u^{(j,k)}\|_{C(\overline{\mathbf{D}})},$$

and $\overline{\mathbf{D}}$ is the closure of the set $\mathbf{D}$.

Problem (1), (2) was studied in [1–3]. The Dirichlet problem for higher order linear hyperbolic equations in a rectangular domain was studied in [4].

Along with problem (1), (2) consider its corresponding homogeneous problem

$$u^{(2,2)}=p_1(x_1,x_2)u^{(2,0)}+p_2(x_1,x_2)u^{(0,2)}+\sum_{j=0}^{1}\sum_{k=1}^{1}p_{jk}(x_1,x_2)u^{(j,k)}, \tag{1_0}$$

$$u(\eta_i(x_2),x_2)=0\ \ (i=1,2);\quad u^{(2,0)}(x_1,\gamma_i(x_1))=0\ \ (i=1,2). \tag{2_0}$$

By a solution of problem (1), (2) we understand a *classical* solution, i.e., a function $u\in C^{2,2}(\mathbf{D})\cap C^{2,0}(\overline{\mathbf{D}})$ satisfying equation (1) and boundary conditions (2) everywhere in $\mathbf{D}$ and $\partial\mathbf{D}$, respectively.

Theorem 1. *Let* $p_i \in C(\overline{\Omega})$ $(i = 1, 2)$, $p_{jk} \in C(\overline{\Omega})$ $(j, k = 0, 1)$, $q \in C(\overline{\Omega})$, $\phi_i \in C^2([0, \omega_2])$, $\psi_i \in C^2([0, \omega_1])$ $(i = 1, 2)$, *and let*

$$p_1(x_1, x_2) \geq 0, \quad p_2(x_1, x_2) \geq 0 \ \text{ for } \ (x_1, x_2) \in \mathbf{D}.$$

Then problem (1), (2) *has the Fredholm property, i.e.:*

(i) *problem* $(1_0), (2_0)$ *has a finite dimensional space of solutions;*

(ii) *problem* (1), (2) *is uniquely solvable if and only if problem* $(1_0), (2_0)$ *has only the trivial solution.*

Furthermore, every solution of problem (1), (2) *in a unique way can be continued to a solution of equation* (1) *in the domain* Ω.

Remark 1. Orthogonal convexity of the domain D is very important and cannot be relaxed. Indeed, in the domain

$$D = \big\{(x_1, x_2) : \ x_1 \in (0, 4), \ x_2 \in (\gamma(x_1), 2)\big\},$$

where

$$\gamma(x) = \begin{cases} e^{\frac{1}{(x-1)(x-3)}} & \text{for } \ x \in (1, 3) \\ 0 & \text{for } \ x \in [0, 1] \cup [3, 4] \end{cases},$$

consider the problem

$$u^{(2,2)} = 0, \tag{3}$$

$$u\big|_{\partial\mathbf{D}} = 0; \quad u^{(2,0)}\big|_{\partial\mathbf{D}} = 1. \tag{4}$$

Notice that the function $y = \gamma(x)$ belongs to $C^\infty([0, 4])$, it is increasing on the interval $[1, 2]$ and it is decreasing on the interval $[2, 3]$. It is easy to show that

$$\eta_1(y) = 2 - \sqrt{1 + \ln^{-1}(y)}$$

is the function inverse to $\gamma(x)$ on the interval $[1, 2]$, and

$$\eta_2(y) = 2 + \sqrt{1 + \ln^{-1}(y)}$$

is the function inverse to $\gamma(x)$ on the interval $[2, 3]$.

It is clear that the only possible solution of problem (3), (4) is a solution of the problem

$$u^{(2,0)} = 1, \tag{5}$$

$$u\big|_{\partial\mathbf{D}} = 0. \tag{6}$$

Problem (5), (6) has the unique solution

$$u(x_1, x_2) = \begin{cases} \dfrac{x_1(x_1 - \eta_1(x_2))}{2} & \text{for } \ x_1 \in [0, 2), \ \ x_2 \in [0, e^{-1}) \\ \dfrac{(x_1 - \eta_2(x_2))(x_1 - 4)}{2} & \text{for } \ x_1 \in (2, 4], \ \ x_2 \in [0, e^{-1}) \\ \dfrac{x_1(x_1 - 4)}{2} & \text{for } \ x_1 \in [0, 4], \ \ x_2 \in (e^{-1}, 2] \end{cases}.$$

One can easily see that $u(x_1, x_2)$ is not a classical solution of problem (3), (4), since it is discontinuous along the line segment $0 \leq x_1 \leq 4$, $x_2 = e^{-1}$.

Remark 2. C^2 smoothness of the boundary of the domain $\mathbf{D}$ is very important and cannot be relaxed. Indeed, let $\alpha \in [1,2)$ be an arbitrary number,

$$\gamma_i(x_2) = 1 + (-1)^i \sqrt{1 - |x_2 - 1|^\alpha} (i = 1,2)$$

and

$$\eta_i(x_1) = 1 + (-1)^i x_1^{\frac{1}{\alpha}} (2 - x_1)^{\frac{1}{\alpha}} \ \ (i = 1,2).$$

In the domain

$$\begin{aligned} \mathbf{D} &= \Big\{ (x_1, x_2) : \ x_1 \in (0,2), \ x_2 \in \big(1 - x_1^{\frac{1}{\alpha}} (2 - x_1)^{\frac{1}{\alpha}}, 1 + x_1^{\frac{1}{\alpha}} (2 - x_1)^{\frac{1}{\alpha}} \big) \Big\} \\ &= \Big\{ (x_1, x_2) : \ x_2 \in (0,2), \ x_1 \in \big(1 - \sqrt{1 - |x_2 - 1|^\alpha}, 1 + \sqrt{1 - |x_2 - 1|^\alpha} \big) \Big\} \end{aligned}$$

consider the problem

$$u^{(2,2)} = 0, \tag{7}$$

$$u(\eta_i(x_2), x_2) = 0 \ \ (i = 1,2); \quad u^{(2,0)}(x_1, \gamma_i(x_1)) = 2 \ \ (i = 1,2). \tag{8}$$

It is clear that the only possible solution of problem (7), (8) is a solution of the problem

$$u^{(2,0)} = 2, \tag{9}$$

$$u(\eta_i(x_2), x_2) = 0 \ \ (i = 1,2). \tag{10}$$

Problem (9), (10) has the unique solution

$$\begin{aligned} u(x_1, x_2) = \big(x_1 - 1 - \sqrt{1 - |x_2 - 1|^\alpha} \big) \big(x_1 - 1 + \sqrt{1 - |x_2 - 1|^\alpha} \big) \\ = (x_1 - 1)^2 - 1 + |x_2 - 1|^\alpha = x_1^2 - 2x_1 + |x_2 - 1|^\alpha. \end{aligned}$$

However, $u^{(0,2)}(x_1, x_2)$ is discontinuous along the line segment $0 \le x_1 \le 2$, $x_2 = 1$, since $\alpha \in [1,2)$. Thus, problem (7), (8) is not solvable in classical sense due to the fact that the boundary $\partial \mathbf{D}$ is not C^2 smooth at points $(0,1)$ and $(2,1)$.

Consider the quasilinear equation

$$\begin{aligned} u^{(2,2)} = \rho_1 \big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)} \big) u^{(2,0)} + \rho_2 \big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)} \big) u^{(0,2)} \\ + \sum_{j=0}^{1} \sum_{k=0}^{1} \rho_{jk} \big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)} \big) u^{(j,k)} + q \big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)} \big), \end{aligned} \tag{11}$$

where $\rho_i(x_1, x_2, \mathbf{z})$ $(i = 1,2)$, $\rho_{jk}(x_1, x_2, \mathbf{z})$ $(j,k = 0,1)$ and $q(x_1, x_2, \mathbf{z})$ are continuous functions on $\overline{\mathbf{D}} \times \mathbb{R}^4$, and $\mathbf{z} = (z_1, z_2, z_3, z_4)$.

Theorem 2. *Let* $\rho_i \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$ $(i = 1,2)$, $\rho_{jk} \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$ $(j,k = 0,1)$, $q \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$, $\phi_i \in C^2([0,\omega_2])$, $\psi_i \in C^2([0,\omega_1])$ $(i = 1,2)$, *and let there exist functions* $P_{il} \in C(\overline{\mathbf{D}})$ $(i,l = 1,2)$ *and* $P_{ijk} \in C(\overline{\mathbf{D}})$ $(i,j = 0,1;\ j,k = 0,1)$ *such that:*

(A_0)

$$0 \le P_{1l}(x_1, x_2) \le \rho_l(x, y, \mathbf{z}) \le P_{2l}(x_1, x_2) \ \ \textit{for} \ \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \quad (l = 1,2);$$

(A_1)

$$P_{1jk}(x_1, x_2) \le \rho_{jk}(x_1, x_2, \mathbf{z}) \le \ P_{2jk}(x_1, x_2) \ \ \textit{for} \ \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \quad (j,k = 0,1);$$

(A_2) *for arbitrary measurable functions* $p_i : \overline{\mathbf{D}} \to \mathbb{R}$ $(i = 1, 2)$ *and* $p_{jk} : \overline{\mathbf{D}} \to \mathbb{R}$ $(j, k = 0, 1)$ *satisfying the inequalities*

$$P_{1l}(x_1, x_2) \le p_l(x, y) \le P_{2l}(x_1, x_2) \ \text{ for } \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \ \ (l = 1, 2),$$
$$P_{1jk}(x_1, x_2) \le p_{jk}(x_1, x_2) \le \ P_{2jk}(x_1, x_2) \ \text{ for } \ (x_1, x_2, \mathbf{z}) \in \overline{\mathbf{D}} \times \mathbb{R}^4 \ \ (j, k = 0, 1),$$

problem $(1_0), (2_0)$ *has only the trivial solution;*

(A_3)

$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{q(x_1, x_2, \mathbf{z})}{\|\mathbf{z}\|} = 0 \quad \textit{uniformly on } \ \overline{\mathbf{D}}.$$

Then problem $(11), (2)$ *has at least one solution.*

Consider the linear and quasilinear equations

$$u^{(2,2)} = \big(p_1(x_1, x_2) u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1, x_2) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2) u + q(x_1, x_2), \tag{12}$$

$$\begin{aligned} u^{(2,2)} = {} & \big(p_1(x_1, x_2, u) u^{(1,0)}\big)^{(1,0)} \\ & + \big(p_2(x_1, x_2, u) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2, u) + q\big(x_1, x_2, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}\big) \end{aligned} \tag{13}$$

and

$$u^{(2,2)} = \big(p_1(x_1, x_2) u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1, x_2) u^{(0,1)}\big)^{(0,1)} + p_0(x_1, x_2, u) + q(x_1, x_2). \tag{14}$$

Theorem 3. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain with* C^2 *boundary inscribed in* Ω *such that*

$$\begin{aligned} \mathbf{D} &= \big\{(x_1, x_2) \in \Omega : \ x_1 \in (0, \omega_1), \ x_2 \in (\gamma_1(x_1), \gamma_2(x_1))\big\} \\ &= \big\{(x_1, x_2) \in \Omega : \ x_2 \in (0, \omega_2), \ x_1 \in (\eta_1(x_2), \eta_2(x_2))\big\}, \end{aligned}$$

where $\gamma_i \in C([0, \omega_1]) \cap C^2((0, \omega_1))$, $\eta_i \in C([0, \omega_2]) \cap C^2((0, \omega_2))$ $(i = 1, 2)$,

$$(-1)^i \gamma_i''(x_1) \le 0 \ \text{ for } \ x_1 \in (0, \omega_1) \ \ (i = 1, 2),$$
$$(-1)^i \eta_i''(x_2) \le 0 \ \text{ for } \ x_2 \in (0, \omega_2) \ \ (i = 1, 2),$$

and

$$\gamma_1(\xi_1^*) = 0, \quad \gamma_2(\xi_2^*) = \omega_2, \quad \eta_1(\zeta_1^*) = 0, \quad \eta_2(\zeta_2^*) = \omega_1$$

for some $\xi_1^*, \xi_2^* \in [0, \omega_1]$ *and* $\zeta_1^*, \zeta_2^* \in [0, \omega_2]$. *Furthermore, let* $p_1 \in C^{1,0}(\overline{\Omega})$, $p_2 \in C^{0,1}(\overline{\Omega})$, $p_0, q \in C(\overline{\Omega})$, $\phi_i \in C^2([0, \omega_2])$, $\psi_i \in C^2([0, \omega_1])$ $(i = 1, 2)$, *and let*

$$p_1(x_1, x_2) \ge 0, \quad p_2(x_1, x_2) \ge 0, \quad p_0(x_1, x_2) \le 0 \ \text{ for } \ (x_1, x_2) \in \mathbf{D}.$$

Then problem $(12), (2)$ *is uniquely solvable, and its solution in a unique way can be continued to a solution of equation* (12) *in the domain* Ω.

Furthermore, if

$$(-1)^i \gamma_i''(x_1) < 0 \ \text{ for } \ x_1 \in (0, \omega_1) \ \ (i = 1, 2) \tag{15}$$

and

$$(-1)^i \eta_i''(x_2) < 0 \ \text{ for } \ x_2 \in (0, \omega_2) \ \ (i = 1, 2), \tag{16}$$

then the solution of problem $(12), (2)$ *can be continued to a solution of equation* (12) *in the closed domain* $\overline{\Omega}$.

Theorem 4. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain same as in Theorem 3, and let* $p_1 \in C^{1,0,1}(\overline{\mathbf{D}} \times \mathbb{R})$, $p_2 \in C^{0,1,1}(\overline{\mathbf{D}} \times \mathbb{R})$, $p_0 \in C(\overline{\mathbf{D}} \times \mathbb{R})$, $q \in C(\overline{\mathbf{D}} \times \mathbb{R}^4)$, *and a nonnegative number* M *be such that*

$$p_1(x_1, x_2, z) \geq 0, \quad p_2(x_1, x_2, z) \geq 0 \ \text{ for } \ (x_1, x_2, z) \in \overline{\mathbf{D}} \times \mathbb{R},$$
$$p_0(x_1, x_2, z) z \leq M \ \text{ for } \ (x_1, x_2, z) \in \overline{\mathbf{D}} \times \mathbb{R},$$
$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{q(x_1, x_2, \mathbf{z})}{\|\mathbf{z}\|} = 0 \ \text{ uniformly on } \ \overline{\mathbf{D}}.$$

Then problem (13), (2) *has at least one solution. Moreover, if inequalities* (15) *and* (16) *hold, then every solution of problem* (13), (2) *belongs to* $C^{2,2}(\overline{\mathbf{D}})$.

Corollary 1. *Let* $\mathbf{D}$ *be an open* ***convex*** *domain same as in Theorem 3, let* $p_1 \in C^{1,0}(\overline{\mathbf{D}})$, $p_2 \in C^{0,1}(\overline{\mathbf{D}})$, $p_0 \in C(\overline{\mathbf{D}} \times \mathbb{R})$, $q \in C(\overline{\mathbf{D}})$, *and let*

$$\big(p_0(x_1, x_2, z_1) - p_0(x_1, x_2, z_1)\big)(z_1 - z_2) \leq 0 \ \text{ for } \ (x_1, x_2, z) \in \overline{\mathbf{D}} \times \mathbb{R}.$$

Then problem (14), (2) *has one and only one solution. Moreover, if inequalities* (15) *and* (16) *hold, then the solution of problem* (13), (2) *belongs to* $C^{2,2}(\overline{\mathbf{D}})$.

Remark 3. Under the conditions of Theorem 3 the functions p_0, p_1 and p_2 may have arbitrary growth order with respect to the phase variable. As an example, consider the equation

$$\begin{aligned} u^{(2,2)} &= \big(e^{\alpha_1(x_1,x_2)u^2} u^{(1,0)}\big)^{(1,0)} + \big(e^{\alpha_2(x_1,x_2)u^3} u^{(0,1)}\big)^{(0,1)} - u^{2n+1} \\ &\quad + \sum_{k=0}^{2n} \beta_k(x_1, x_2) u^k + \Big(1 + |u| + |u^{(1,0)}| + |u^{(0,1)}| + |u^{(1,1)}|\Big)^{1-\varepsilon}, \end{aligned} \tag{17}$$

where $\alpha_1 \in C^{1,0}(\overline{\mathbf{D}})$, $\alpha_2 \in C^{0,1}(\overline{\mathbf{D}})$, $\beta_k \in C(\overline{\mathbf{D}})$ $(k = 0, \dots, 2n)$ are arbitrary functions, n is an arbitrary positive integer, and $\varepsilon \in (0, 1)$. By Theorem 4, problem (17), (2) has at least one solution.

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A Bayesian Optimization Approach for Selecting the Best Parameters for Weighted Finite Difference Scheme Corresponding to Heat Equation

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In the domain $[0;1]\times[0;T]$, where $T = const > 0$, let us consider the initial-boundary value problem for the heat equation

$$
\begin{gathered}
\frac{\partial U(x,t)}{\partial t} - a\frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t),\\
U(0,t) = U(1,t) = 0, \ \ t \geq 0,\\
U(x,0) = U_0(x), \ \ x \in [0;1],
\end{gathered} \tag{1}
$$

where a is a positive constant and U_0 and f are given functions.

For the numerical solution of problem (1) let us introduce a net whose mesh points are denoted by $(x_i,t_j) = (ih, j\tau)$, where $i = 0,1,\dots,M$; $j = 0,1,\dots,N$ with $h = 1/M$, $\tau = T/N$ and consider the following weighted finite difference scheme (see, for example, [8]):

$$
\begin{gathered}
\frac{u_i^{j+1} - u_i^j}{\tau} - a\Big[\sigma_1\,\frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + \sigma_2\,\frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{h^2}\Big] = \eta_1 f_i^{j+1} + \eta_2 f_i^j,\\
i = 1,2,\dots,M-1; \ \ j = 0,1,\dots,N-1,\\
u_0^j = u_M^j = 0, \ \ j = 0,1,\dots,N,\\
u_i^0 = U_{0,i}, \ \ i = 0,1,\dots,M.
\end{gathered} \tag{2}
$$

Here the initial line is denoted by $j = 0$. The discrete approximation at (x_i,t_j) is denoted by u_i^j and the exact solution to problem (1) at those points is denoted by U_i^j.

Qualitative and quantitative properties, as well as numerical solution for problem (1) and its nonlinear analogs are well studied in the literature (see, for example, [2, 3, 8] and the references therein). By tuning the parameters τ, h, σ_1, σ_2, η_1, η_2 and take relevant approximation for the right side the stability of the scheme (2), the different accuracy can be achieved for the numerical solution.

Our goal is to find the above-mentioned parameters automatically by using Bayesian machine learning. In particular, we will minimize objective function applying Bayesian Optimization (BO). The objective function is designed as a maximum of the absolute value of the difference between exact and numerical solutions at each grid point (x_i,t_j), $i = 0,1,\dots,M$; $j = 0,1,\dots,N$. For training, the different types of initial and boundary conditions with the corresponding right-hand side were selected. The output of the objective function depend on unknown parameters implicitly. Thus, we deal with, so-called black-box function optimization problem [1]. Since we do not have the close formula for the objective function, there is no information regarding gradient. So, the derivative-free optimization method is needed. BO is one of the most popular black-box optimization methods [1, 4–6]. It is based on Gaussian Process (GP) and Bayes Theorem [7]. BO is a model-based approach that makes sequential decisions to search the space, so the number of simulations gets minimized.

A GP is a generalization of the Gaussian Probability Distribution. Notation for Gaussian probability distribution is $\mathcal{N}(\mu,\sigma)$, where μ is mean and σ is standard deviation of random variables. While a Gaussian probability distribution describes random variables which are scalars or vectors, a stochastic process governs the properties of functions. GP is an extension of Multivariate Gaussian Distribution. In turn, the multivariate Gaussian distribution is a generalization of the one-dimensional normal distribution to higher dimensions. The probability density function of the multivariate Gaussian distribution in D-dimensions is defined by the following formula:

$$g(z)=\frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}}\exp\Big[-\frac{1}{2}\,(z-\mu)'\Sigma^{-1}(z-\mu)\Big],$$

where, in general Σ' denotes transpose of Σ, Σ^{-1} denotes inverse of Σ, $\mu=(\mu_1,\mu_2,\dots,\mu_D)$ is mean vector of $z=(z_1,z_2,\dots,z_D)$ and $\Sigma=\operatorname{cov}[z]$ is the $D\times D$ covariance matrix, which is positively defined and is constructed by, one of the so-called covariance functions [7]. One of the common covariance function is Squared Exponential function:

$$K_{SE}=k(z_i,z_j)=\sigma\exp\Big(-\frac{1}{2}\,\frac{\|z_i-z_j\|^2}{\theta^2}\Big),$$

where σ and θ are hyper-parameters which can be tuned by users. Note that GP is fully described by mean and covariance functions.

Most of the efficiency derived from Bayesian optimization ability to incorporate prior belief about the problem to help direct the sampling, and to trade of exploration and exploitation of the search space [1]. Algorithm is called Bayesian because it uses the well-known Bayes Theorem, which can be stated as follows

$$P(A\mid B)\sim P(B\mid A)P(A),$$

where $P(A\mid B)$ is probability of A given B, $P(B\mid A)$ is probability of B given A and $P(A)$ is the marginal probability [1, 7].

Let us now consider how the new query can be obtained using the aforementioned Bayes Theorem. Assume, the dataset with n points is already obtained $D_{1:n}=\{z_{1:n},g(z_{1:n})\}$. Bayes Theorem helps to estimate posterior distribution $P(g\mid D_{1:n})$ by combining a prior distribution $P(g)$ with the likelihood function $P(D_{1:n}\mid g)$

$$P(g\mid D_{1:n})\sim P(D_{1:n}\mid g)P(g).$$

To find the next sample point z_{n+1}, the so-called acquisition function is maximized. There are different types of acquisition functions. One of the most popular acquisition function is Upper Confidence Bound (UCB)

$$\mathrm{UCB}\,(z)=\mu(z)+\kappa\sigma(z),$$

where κ is tunable trade-off parameter.

The BO algorithm performs as follows:

1. Collect data $D_{1:n}=\{z_{1:n},g(z_{1:n})\}$ and fit the GP. Note, that BO can be started from one point dataset;
2. Find the next querying point by maximizing acquisition function;
3. Augment dataset $D_{1:n+1}=\{z_{1:n+1},g(z_{1:n+1})\}$ and update GP;

4. End process when the desired accuracy is obtained or the number of iterations reaches a certain value.

Note that all steps in the BO algorithm are clear except step 3 (note also that since the evaluation of the acquisition function is not expensive its maximization in step 2 can be done by some standard optimization algorithm). In step 3 we need to update the GP and find the updated mean and variance functions, based on which the acquisition functions are constructed. Bellow, the close formulas for calculating the updated mean and variance functions are given. Assume, dataset $D_{1:n} = \{z_{1:n}, g(z_{1:n})\}$ is already obtained. The function values are drawn according to a multivariate normal distribution $\mathcal{N}(0, K)$, where the kernel matrix is given by:

$$K = \begin{bmatrix} k(z_1, z_1) & \cdots & k(z_1, z_n) \\ \vdots & \ddots & \vdots \\ k(z_n, z_1) & \cdots & k(z_n, z_n) \end{bmatrix}.$$

Let us denote $g_{n+1} = g(z_{n+1})$, where z_{n+1} is the next sampling point, which is obtained from the maximization of the acquisition function. g_{n+1} and $g_{1:n}$ are jointly Gaussian:

$$\begin{bmatrix} g_{1:n} \\ g_{n+1} \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} K & k \\ k' & k(z_{n+1}, z_{n+1}) \end{bmatrix}\right),$$

where

$$k = \big[k(z_{n+1}, z_1), k(z_{n+1}, z_2), \dots, k(z_{n+1}, z_n)\big].$$

Using the Sherman–Morrison–Woodbury formula [1,7] the following predictive distribution can be obtained:

$$P(g_{n+1} \mid D_{1:n+1}, z_{n+1}) \sim \mathcal{N}\big(\mu_n(z_{n+1}), \sigma_n^2(z_{n+1})\big),$$

where

$$\begin{aligned} \mu_n(z_{n+1}) &= k' K^{-1} g_{1:n}, \\ \sigma_n^2(z_{n+1}) &= k(z_{n+1}, z_{n+1}) - k' K^{-1} k. \end{aligned}$$

To implement the BO for our problem the IMGPO (Infinite-Metric GP Optimization) algorithm is used [4]. Note that the IMGPO algorithm does not require any prior data. It can be started from any random point, say from the center point of the search space, as in our case. IMGPO uses *UCB* acquisition function and avoids its maximization for finding the next sample point, instead it handles the tradeoff with the assumption of the existence of a tighter bound than *UCB* and remain the exponential convergence at the same time (for details see [4]).

In our numerical experiment we took $\sigma_2 = 1 - \sigma_1$, $\eta_2 = 1 - \eta_1$ and the search space is $(\tau, h, \sigma_1, \eta_1) \in [0; 0.1] \times [0; 0.1] \times [0; 1] \times [0; 1]$. The stopping criterion for BO is as follows, the algorithm stops when maximum error is less then $\tau + h^2$ and less than $\varepsilon = 0.0001$ or maximum error is less than $\tau + h^2$ and maximum number of iterations $I = 30$ is reached.

We have carried out various numerical experiments for different test cases and found the values of parameters for the best performance of the scheme (2).

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On the Behavior of Solutions with Positive Initial Data to Third Order Differential Equations with General Power-Law Nonlinearities

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1 Introduction

$$y''' = p(x,y,y',y'')|y|^{k_0}|y'|^{k_1}|y''|^{k_2}\operatorname{sgn}(yy'y''), \quad k_0,k_1,k_2>0, \tag{1.1}$$

with positive continuous and Lipschitz continuous in u, v, w function $p(x,u,v,w)$ satisfying inequalities

$$0 < m \le p(x,u,v,w) \le M < +\infty. \tag{1.2}$$

Equation (1.1) in the case $k_0>0$, $k_0 \neq 1$, $k_1=k_2=0$, was studied by I. Astashova in [1, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [4,6], and proved in [3].

For higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7,8]. Another approach to study asymptotic properties of solutions to higher order equations was offered by I. T. Kiguradze and T. A. Chanturia in [9].

Using methods described in [1,2,5] by I. V. Astashova, the behavior of solutions to (1.1) near domain boundaries is considered with respect to the values k_0, k_1 and k_2.

2 Main results

Consider positive increasing convex solutions to equation (1.1).

Theorem 2.1. *Suppose the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*, and let $y(x)$ be a positive increasing convex on (x_1,x_2) solution to equation* (1.1)*. Then for $k_2 \neq 2$ the following estimates hold:*

$$m(y(x_1))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{2-k_2}}{2-k_2}\bigg|_{x_1}^{x_2} \le M(y(x_2))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2}, \tag{2.1}$$

and for $k_2 \neq 1$ the following estimates hold:

$$m(y'(x_1))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{1-k_2}}{1-k_2}\bigg|_{x_1}^{x_2} \le M(y'(x_2))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2}. \tag{2.2}$$

Proof. Let us prove inequalities (2.1). Since $y(x)$ is positive, increasing and convex, for $x \in [x_1,x_2]$ we have

$$m(y(x_1))^{k_0}(y'(x))^{k_1}(y''(x))^{k_2} \le y''' \le M(y(x_2))^{k_0}(y'(x))^{k_1}(y''(x))^{k_2},$$

hence

$$m(y(x_1))^{k_0}(y'(x))^{k_1}y'' \le y'''(y''(x))^{1-k_2} \le M(y(x_2))^{k_0}(y'(x))^{k_1}y''.$$

Let us integrate the above inequality on $[x_1, x_2]$:

$$m(y(x_1))^{k_0}\int\limits_{x_1}^{x_2}(y'(x))^{k_1}\,dy' \le \int\limits_{x_1}^{x_2}(y''(x))^{1-k_2}\,dy'' \le M(y(x_2))^{k_0}\int\limits_{x_1}^{x_2}(y'(x))^{k_1}\,dy',$$

so

$$m(y(x_1))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{2-k_2}}{2-k_2}\bigg|_{x_1}^{x_2} \le M(y(x_2))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2},$$

and thus, estimates (2.1) are obtained.

Now let us prove inequalities (2.2). Due to equation (1.1) and that fact that the function $p(x,u,v,w)$ is bounded, for any $x \in [x_1, x_2]$ it holds that

$$m(y'(x_1))^{k_1-1}(y(x))^{k_0}y'(x)(y''(x))^{k_2} \le y''' \le M(y'(x_2))^{k_1-1}(y(x))^{k_0}y'(x)(y''(x))^{k_2},$$

and therefore

$$m(y'(x_1))^{k_1-1}(y(x))^{k_0}y'(x) \le y'''(y''(x))^{-k_2} \le M(y'(x_2))^{k_1-1}(y(x))^{k_0}y'(x).$$

By integrating these inequalities on $[x_1, x_2]$, we obtain

$$m(y'(x_1))^{k_1-1}\int\limits_{x_1}^{x_2}(y(x))^{k_0}\,dy \le \int\limits_{x_1}^{x_2}(y''(x))^{-k_2}\,dy'' \le M(y'(x_2))^{k_1-1}\int\limits_{x_1}^{x_2}(y(x))^{k_0}\,dy,$$

which implies

$$m(y'(x_1))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{1-k_2}}{1-k_2}\bigg|_{x_1}^{x_2} \le M(y'(x_2))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2},$$

and estimates (2.2) are also proved. □

Theorem 2.2. *Suppose the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*. Then the second derivative of any maximally extended solution $y(x)$ to equation* (1.1)*, satisfying the conditions $y(x_0) = y_0 > 0$, $y'(x_0) = y_1 > 0$, $y''(x_0) = y_2 > 0$ at some point x_0, tends to $+\infty$ as $x \to \widetilde{x}$, where $\widetilde{x}$ is the right domain boundary of solution $y(x)$, $x_0 < \widetilde{x} \le +\infty$.*

Proof. Since initial data are positive and $p(x,u,v,w) > m$, we obtain $y'''(x) \ge my_0^{k_0}y_1^{k_1}y_2^{k_2}$ for $x \ge x_0$.

Denote $C_0 = my_0^{k_0}y_1^{k_1}y_2^{k_2}$, then $y''' \ge C_0$, and by consequently integrating obtained inequalities on $[x_0, x]$ we derive

$$y''(x) > C_0(x-x_0), \quad y'(x) > \frac{C_0}{2}\,(x-x_0)^2, \quad y(x) > \frac{C_0}{6}\,(x-x_0)^3.$$

Then from equation (1.1) it follows that

$$y'''(x) > m\Big(\frac{C_0}{6}\,(x-x_0)^3\Big)^{k_0}\Big(\frac{C_0}{2}\,(x-x_0)^2\Big)^{k_1}(C_0(x-x_0))^{k_2} = \frac{mC_0^{k_0+k_1+k_2}}{6^{k_0}2^{k_1}}\,(x-x_0)^{3k_0+2k_1+k_2},$$

that is,

$$y''(x) > \widetilde{C}_0(x-x_0)^{3k_0+2k_1+k_2+1},$$

where $\widetilde{C}_0 > 0$ is a constant. Thus, $y''(x) \to +\infty$ as $x \to +\infty$, and the theorem is proved for $\widetilde{x} = +\infty$.

Consider now the case $\widetilde{x} < +\infty$. If for a constant $D > 0$ inequality $y''(x) \le D$ holds for $x \in (x_0, \widetilde{x})$, then

$$\begin{gathered}
y'(x) \le D(x-x_0) + y'(x_0) \le D(\widetilde{x}-x_0) + y'(x_0) = D_1 < +\infty, \\
y(x) \le D_1(x-x_0) + y(x_0) \le D_1(\widetilde{x}-x_0) + y'(x_0) = D_2 < +\infty,
\end{gathered}$$

so $y'''(x) \le MD_2^{k_0}D_1^{k_1}D^{k_2} < +\infty$, and, since the solution and all its derivatives up to the third are increasing and bounded on a finite interval, there exist finite limits of the solution and its derivatives as $x \to \widetilde{x}$. Then the solution $y(x)$ can be extended to the right of $\widetilde{x}$, and we obtain a contradiction.

Thus, $y''(x) \to +\infty$ as $x \to \widetilde{x}$, and the theorem is proved. □

Theorem 2.3. *Suppose $k_0 + k_1 + k_2 > 1$, and the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*. Then for any maximally extended solution $y(x)$ to equation* (1.1)*, satisfying the conditions $y(x_0) \ge 0$, $y'(x_0) \ge 0$, $y''(x_0) = y_2 > 0$ at some point x_0, its right domain boundary $\widetilde{x}$ is finite and satisfies the estimate*

$$\widetilde{x} - x_0 < \xi y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}},$$

with $\xi = \big(\frac{(2k_0+k_1+1)2^{k_0}}{m}\big)^{\frac{1}{2k_0+k_1+1}}\big(1 - 2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}\big)^{-1}$.

Proof. As it was shown above, the second derivative of such solution is infinitely increasing as argument tends to the right domain boundary. Consider the sequence of points x_i, $i = 0, 1, \dots$, such that $y''(x_i) = 2y''(x_{i-1}) = 2^i y_2$.

For $x \in [x_i, x_{i+1}]$ the following inequalities hold:

$$\begin{gathered}
y''(x) \ge 2^i y_2, \\
y'(x) > y'(x) - y'(x_i) \ge 2^i y_2(x-x_i), \\
y(x) > y(x) - y(x_i) \ge 2^{i-1}\, y_2(x-x_i)^2.
\end{gathered}$$

Then from equation (1.1) we derive

$$\begin{gathered}
y'''(x) > m\big|2^{i-1}y_2(x-x_i)^2\big|^{k_0}\big|2^i y_2(x-x_i)\big|^{k_1}|2^i y_2|^{k_2}, \\
y'''(x) > m \cdot 2^{i(k_0+k_1+k_2)-k_0} y_2^{k_0+k_1+k_2}(x-x_i)^{2k_0+k_1}.
\end{gathered}$$

By integrating this inequality on $[x_i, x_{i+1}]$, we obtain

$$\begin{gathered}
2^{i+1}y_2 - 2^i y_2 > \frac{m \cdot 2^{i(k_0+k_1+k_2)-k_0}}{2k_0+k_1+1}\, y_2^{k_0+k_1+k_2}(x_{i+1}-x_i)^{2k_0+k_1+1}, \\
2^i y_2^{-(k_0+k_1+k_2-1)} > \frac{m \cdot 2^{i(k_0+k_1+k_2)-k_0}}{2k_0+k_1+1}\,(x_{i+1}-x_i)^{2k_0+k_1+1}, \\
(x_{i+1}-x_i)^{2k_0+k_1+1} < \frac{(2k_0+k_1+1)\cdot 2^{k_0}}{m}\,(2^i y_2)^{-(k_0+k_1+k_2-1)},
\end{gathered}$$

and, since $2k_0 + k_1 + 1 > 0$,

$$x_{i+1} - x_i < \Big(\frac{(2k_0 + k_1 + 1)2^{k_0}}{m}\Big)^{\frac{1}{2k_0+k_1+1}} (2^i y_2)^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Now let us summarize these inequalities:

$$\sum_{i=0}^{+\infty}(x_{i+1} - x_i) < \Big(\frac{(2k_0 + k_1 + 1)2^{k_0}}{m}\Big)^{\frac{1}{2k_0+k_1+1}} y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}} \sum_{i=0}^{+\infty} 2^{-i\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Since $k_0 + k_1 + k_2 > 1$, the series in the right part converges and

$$\widetilde{x} - x_0 = \lim_{i\to+\infty} x_i - x_0 = \sum_{i=0}^{+\infty}(x_{i+1} - x_i) < \xi y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}},$$

with $\xi = \big(\frac{(2k_0+k_1+1)2^{k_0}}{m}\big)^{\frac{1}{2k_0+k_1+1}}(1 - 2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}})^{-1}$.

Thus, $\widetilde{x}$ is finite and the theorem is proved. □

Theorem 2.4. *Suppose $k_0+k_1+k_2 \neq 1$, $k_2 \neq 1$, $k_2 \neq 2$, and the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*. Let $y(x)$ be a maximally extended solution to equation* (1.1)*, satisfying the conditions $y(x_0) \geq 0$, $y'(x_0) \geq 0$, $y''(x_0) > 0$ at some point x_0. Then*

1. *if $k_0 + k_1 + k_2 < 1$, then $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < +\infty$ or $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} = +\infty$;*
2. *if $k_0 + k_1 + k_2 > 1$, $k_1 \leq 1$, $k_2 < 1$, then $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
3. *if $k_1 > 1$, $k_2 < 1$, then $y \to const$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$ or $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
4. *if $1 < k_2 < 2$, then $y \to const$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
5. *if $k_2 > 2$, then $y \to const$, $y' \to const$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$.*

Proof. Since the initial data are nonnegative as well as the function $p(x,u,v,w)$, solution $y(x)$ and its first, second and third derivatives are positive and increasing as $x \to \widetilde{x}$, where $\widetilde{x}$ is a right domain boundary of $y(x)$. According to the Theorem 2.2, the second derivative is increasing and unbounded.

Let us show that if the first derivative is bounded, then the solution with positive initial data cannot be bounded. Indeed, let $y' \leq C$, then $y \leq C(x - x_0) + y(x_0)$, which implies that in the case $\widetilde{x} < +\infty$ the solution is also bounded. If the solution is infinitely extensible to the right, then, since $y'(x_0) > 0$, we derive $y(x) > y'(x_0)(x - x_0) + y(x_0)$, and unboundedness of this solution follows from unboundedness of x.

Thus, there are three possible options: a solution and its first derivative are bounded; a solution is bounded, but its derivative is unbounded, and both solution and its derivative are unbounded.

At first, let us consider the case $k_2 > 2$. In this case $k_0 + k_1 + k_2 > 1$, and by Theorem 2.3, the domain of solution is finite. Values $k_0 + 1$ and $k_1 + 1$ are positive; besides, $1 - k_2 < 2 - k_2 < 0$, and therefore, using inequality (2.1) on the interval (x_0, x), as $x \to \widetilde{x}$ we have

$$m(y(x_0))^{k_0} \frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1 + 1} \leq \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2 - k_2} < +\infty,$$

which implies that $y'(x)$ is bounded as $x \to \widetilde{x}$. Analogously, inequality (2.1) implies that the solution $y(x)$ is also bounded.

Consider the case $1 < k_1 < 2$. Again, $k_0 + k_1 + k_2 > 1$, and, by Theorem 2.3, the domain of $y(x)$ is finite; also $1 - k_2 < 0 < 2 - k_2$, and, due to (2.1), (2.2) and the fact that $y'' \to +\infty$ as $x \to \widetilde{x}$, we derive

$$M(y(x))^{k_0}\,\frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1+1} \ge \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2-k_2} \to +\infty,$$
$$m(y'(x_0))^{k_1-1}\,\frac{(y(x))^{k_0+1} - (y(x_0))^{k_0+1}}{k_0+1} \le \frac{(y''(x))^{1-k_2} - (y''(x_0))^{1-k_2}}{1-k_2} < +\infty,$$

hence $y(x) \to const$ and $y'(x) \to +\infty$ as $x \to \widetilde{x}$.

Further, suppose $k_2 < 1$, $k_1 > 1$. Then $k_0 + k_1 + k_2 > 1$, the domain of solution is finite, $2 - k_2 > 1 - k_2 > 0$, and we obtain

$$M(y(x))^{k_0}\,\frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1+1} \ge \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2-k_2} \longrightarrow +\infty,$$
$$M(y'(x))^{k_1-1}\,\frac{(y(x))^{k_0+1} - (y(x_0))^{k_0+1}}{k_0+1} \ge \frac{(y''(x))^{1-k_2} - (y''(x_0))^{1-k_2}}{1-k_2} \longrightarrow +\infty.$$

In this case there are two possible options: $y \to const$, $y' \to +\infty$, and $y \to +\infty$, $y' \to +\infty$.

Finally, for $k_2 < 1$, $k_1 \le 1$, according to the above inequalities, the only possible option is $y \to +\infty$, $y' \to +\infty$; moreover, if $k_0 + k_1 + k_2 > 1$, then $\widetilde{x} < +\infty$, and the theorem is proved. $\square$

Remark 2.1. In the cases 1 and 3 Theorem 2.4 does not state the existence of solutions of every possible type of behavior. In the cases 4 and 5 for $k_0 \ge 1$, $k_1 \ge 1$, $k_2 > 1$ the existence of solutions of described type is guaranteed by classical existence and uniqueness theorem. For $0 < k_0 < 1$, $k_1 \ge 1$, $k_2 \ge 1$ the existence of solutions to equation (1.1) with positive initial data is guaranteed by the following theorem.

Theorem 2.5 (I. Astashova, [1])**.** *Let the function $p(x, y_0, \dots, y_{n-1})$ be continuous in x and Lipschitz continuous in $y_0, \dots, y_{n-1}$. Then for any set of numbers $x_0, y_0^0, \dots, y_{n-1}^0$ with not every y_i^0 equal to zero, the corresponding Cauchy problem for the equation*

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y, \quad n \ge 2, \quad 0 < k < 1,$$

has a unique solution.

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Solution of Izobov–Bogdanov Problem on Irregularity Sets of Linear Differential Systems with a Parameter-Multiplier

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We consider depending on a parameter $\mu \in \mathbb{R}$ linear differential system

$$\dot{x} = \mu C(t)x, \ \ x(t) \in \mathbb{R}^n, \ \ t \geq 0 \tag{1_μ}$$

with a piecewise continuous bounded coefficients. By an irregularity set of the system

$$\dot{x} = C(t)x, \ \ x(t) \in \mathbb{R}^n, \ \ t \geq 0 \tag{2_C}$$

we call [2] the set of those values $\mu \in \mathbb{R}$ such that the corresponding system (1_μ) is irregular under Lyapunov.

E. K. Makarov constructed (see references in [2]) examples of systems (2_C) that have various metric and topological properties of their irregularity sets. Some of them have an arbitrary Lebesgue measure [5].

Later E. A.Barabanov proved [1] that every open set of real line without zero point can be realized as irregularity set of some system (2_C). Paper [4] held an analogous result for closed sets.

Recently P. A. Khudyakova has established that the reducibility sets of systems (1_μ) are exactly the class of F_σ sets [3].

In the present talk we completely describe the structure of irregularity sets for system (2_C), that solve N. A. Izobov's problem from [2].

For every $\varphi \in \mathbb{R}$ we denote a rotation matrix with the angle φ clockwise as

$$U(\varphi) \equiv \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

and let

$$J := U(2^{-1}\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For each $y = (y_1, y_2)^\top \in \mathbb{R}^2$ and 2×2-matrix Z we use the notations $\|y\| \equiv \sqrt{y_1^2 + y_2^2}$ for an Euclid norm and $\|Z\| \equiv \max\limits_{\|y\|=1} \|Zy\|$ for a spectral norm.

For any strongly increasing sequence $\{m_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ and for the numbers $5 \leq i_k \in \mathbb{N}$ we define the sequence $\{T_k\}_{k=1}^{+\infty}$, setting

$$T_1 := 2, \ \ T_{k+1} := m_k(i_k + 2)T_k, \ \ k \in \mathbb{N}.$$

Next let

$$\theta_k := m_k i_k T_k, \ \ \tau_k := \theta_k + m_k T_k, \ \ k \in \mathbb{N}.$$

For every sequence $\{b_k\}_{k=1}^{+\infty} \subset \mathbb{R}$ and for a number $d \in \mathbb{R}$, $d \neq 0$, we define the matrix $A(\,\cdot\,) = A(\,\cdot\,, d, \{m_k, i_k, b_k\}_{k=1}^{\infty})$, for each $l = \overline{1, T_k}$, $k \in \mathbb{N}$ setting

$$\begin{aligned} A(t) &\equiv b_k J, \quad t \in (\tau_k - m_k l, \tau_k - m_k l + 1], \\ A(t) &\equiv -b_k J, \quad t \in [\tau_k + m_k l - 1, \tau_k + m_k l). \end{aligned}$$

For all other $t \geq 0$ let $A(t) \equiv d \operatorname{diag}[1, -1]$.

We denote as $X_A(t,s)$ the Cauchy matrix for system (2_A) and define the number $\delta(d)$ in the case $d > 0$ by the equality $\delta(d) := 1$, and in the case $d < 0$, let $\delta(d) := 2$. Let us denote as well

$$L_d(\alpha) := \Big\{ x \in \mathbb{R}^2 : \ \Big| \frac{x_{3-\delta(d)}}{x_{\delta(d)}} \Big| \leq \alpha \Big\}.$$

Note that

$$\begin{pmatrix} m & 0 \\ 0 & \dfrac{1}{m} \end{pmatrix} L_d(\alpha) = L_d(m^{-2 \operatorname{sgn} d} \alpha).$$

Lemma 1. *The matrix $X_A(T_{k+1}, \theta_k)$ is self-conjugated.*

For all $d \neq 0$ we define $k_0(d) \in \mathbb{N}$ by the equality $k_0(d) := 2 + [|d|^{-1}]$ ($[\,\cdot\,]$ denotes the integer part of a number).

Lemma 2. *For every $k \in \mathbb{N}$, $k \geq k_0(d) - 1$, the next inclusion holds*

$$X(T_{k+1}, T_{k_0(d)}) e_{\delta(d)} \subset L_d(2e^{4m_k T_k |d|}).$$

Let us denote

$$\widehat{Y}_\varkappa(\gamma) := U(\gamma) \operatorname{diag}[e^{\varkappa}, e^{-\varkappa}], \quad \gamma, \varkappa \in \mathbb{R}.$$

Lemma 3. *For all $\gamma, \varkappa \in \mathbb{R}$ such that $|\cos\gamma| \leq e^{-2|\varkappa|}$, the next estimation is true $\|\widehat{Y}_\varkappa^2(\gamma)\| < e^2$.*

Lemma 4. *If $d \neq 0$ and there exist $l \in \mathbb{N}$ and a sequence $(k_j)_{j=1}^{+\infty} \subset \mathbb{N}$ such that for all $p \in (k_j)_{j=1}^{+\infty}$ both the inequalities $i_p \leq l$, $m_p \geq 2\max\{l, |d|^{-1}\}$ and the estimate $|\cos b_p| < e^{-2m_p|d|}$ hold, then system (2_A) is irregular under Lyapunov.*

Let us denote

$$\widetilde{L}_\varkappa := L_{\operatorname{sgn}\varkappa}(2^3\varkappa^2), \quad \varkappa \in \mathbb{R}, \quad \widehat{L}_{k,d} := L_d(2^3 d^2 (m_k - 1)^2).$$

Lemma 5. *For all $\gamma, \varkappa \in \mathbb{R}$, $|\sin\gamma| \geq \varkappa^{-2}$, $\varkappa > 2^4$, the inclusion*

$$\widehat{Y}_\varkappa\Big(\gamma + \frac{\pi}{2}\Big) \widetilde{L}_\varkappa \subset \widetilde{L}_\varkappa$$

and for any $x \in \widetilde{L}_\varkappa$ the inequality

$$\Big\| \widehat{Y}_\varkappa\Big(\gamma + \frac{\pi}{2}\Big) x \Big\| > \|x\| e^{\varkappa - \sqrt{\varkappa}}$$

are correct.

Lemma 6. *For all $d \neq 0$, $k \in \mathbb{N}$ such that*

$$m_k > 1 + 2^4 |d|^{-1}, \quad |\cos b_k| \geq d^{-2}(m_k - 1)^{-2},$$

the inclusion

$$X_A(T_{k+1}, \theta_k - m_k + 1) \widehat{L}_{k,d} \subset \widehat{L}_{k,d}$$

holds, and for any solution $x(\,\cdot\,)$ of system (2_A) with the initial condition $x(\theta_k - m_k + 1) \in \widehat{L}_{k,d}$ for every $1 \leq l \leq 2T_k$ the next estimation is true

$$\frac{\|x(\theta_k + m_k l)\|}{\|x(\theta_k + m_k(l-1))\|} \geq e^{|d|(m_k - 1) - \sqrt{|d|(m_k - 1)}}.$$

Lemma 7. *If $m_k \to +\infty$ whereas $k \to +\infty$ and for any $l \in \mathbb{N}$ there exists $k_l \in \mathbb{N}$ such that for all $k \geq k_l$, satisfying the condition $i_k \leq l$, the estimate $|\cos b_k| > |d|^{-2}(m_k - 1)^{-2}$ holds, then system (2_A) is regular under Lyapunov.*

Let M be an arbitrary $G_{\delta\sigma}$ set. One can find an open sets $\check{M}_{n,l} \subset \mathbb{R}$, $l, n \in \mathbb{N}$, for which the sets $\widetilde{M}_l$, $l \in \mathbb{N}$, defined by the equalities $\widetilde{M}_l := \bigcap\limits_{n=1}^{+\infty} \check{M}_{n,l}$, satisfy the relation $M = \bigcup\limits_{l=1}^{+\infty} \widetilde{M}_l$. Let us denote $\widehat{M}_{n,l} := \bigcap\limits_{p=1}^{n} \check{M}_{p,l}$. It is easy to see that the inclusion $\widehat{M}_{n+1,l} \subset \widehat{M}_{n,l}$ as well as the equality $\widetilde{M}_l = \bigcap\limits_{n=1}^{+\infty} \widehat{M}_{n,l}$ are correct.

We define by the recurrence a sequence $\{j_n\}_{n=0}^{\infty} \subset \mathbb{N} \cup \{0\}$, by set up

$$j_0 := 0, \ \ j_n := 2n\, 9^{n+n^3} + j_{n-1}, \ \ n \in \mathbb{N}.$$

For any $k, l, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we denote

$$J_n := \{j_{n-1}+1, \dots, j_n\}, \quad \varkappa_k(n) := 9^{-n-n^3}\big(k - 2^{-1}(j_n + j_{n-1})\big),$$
$$\rho_{n,l}(\alpha) = \rho_{n,l}(\alpha, \widehat{M}_{n,l}) := \inf_{\beta \in \mathbb{R} \setminus \widehat{M}_{n,l}} |\alpha - \beta|.$$

Moreover, let us denote $I_{n,k} = I_{n,k}(\{\widehat{M}_{n,l}\}_{n,l \in \mathbb{N}})$ for the set of all $l \in \mathbb{N}$ such that either $\rho_{n,l}(\varkappa_k(n)) \geq 2n^{-1}$, or there exists $p \in \{1, \dots, n-1\}$ for which

$$2n^{-1} \leq \rho_{p,l}(\varkappa_k(n)) \leq 5n^{-1}.$$

Lemma 8. *For all $\mu \notin M$ and $l \in \mathbb{N}$ one can find $n_0 = n_0(\mu, l) \in \mathbb{N}$ such that for every $n \geq n_0$ the correctness for some $k \in J_n$ of the inequality $|\mu - \varkappa_k(n)| < 2n^{-1}$ implies the inclusion $l \notin I_{n,k}$.*

For any integer k there exists a singular $n = n(k) \in \mathbb{N}$, for which $k \in J_n$. We define the values m_k, i_k and b_k, depending on a choice of the open sets $\check{M}_{n,l} \subset \mathbb{R}$, $l, n \in \mathbb{N}$, such that $M = \bigcup\limits_{l=1}^{+\infty} \bigcap\limits_{n=1}^{+\infty} \check{M}_{n,l}$, by the equalities

$$d := \mu, \ \ \mu \in \mathbb{R}, \ \ m_k := 1 + n(k)^2, \ \ n \in \mathbb{N}.$$

Let

$$i_k := \max\big\{5, \min I_{n,k}\big\}, \quad b_k(\mu) := 2^{-1}\pi + n^{-1}(\mu - \varkappa_k(n)), \ \ \mu \in \mathbb{R},$$

in the case $I_{n,k} \neq \varnothing$, and let

$$i_k := 5, \ \ b_k(\mu) \equiv 0, \ \ \text{if} \ \ I_{n,k} = \varnothing.$$

Let us define the matrix $\widetilde{A}_\mu(\,\cdot\,) = \widetilde{A}_\mu(\,\cdot\,, \{\widehat{M}_{n,l}\}_{n,l \in \mathbb{N}})$, $\mu \in \mathbb{R}$, by the equality

$$\widetilde{A}_\mu(t) := A(t) = A(t, d, \{m_k, i_k, b_k\}_{k=1}^{\infty}), \ \ t \geq 0,$$

with the defined as above values of parameters d, m_k, i_k, b_k.

Lemma 9. *If $0 \notin M$, then the system $(2_{\widetilde{A}_\mu})$ is irregular under Lyapunov for all $\mu \in M$ and is regular for any other $\mu \in \mathbb{R} \setminus M$.*

Let us denote by $\mathcal{T}$ the set of all $t \in \mathbb{R}_+ := \mathbb{R} \cap [0, +\infty)$ such that $\widetilde{A}_\mu(t) = \mu \operatorname{diag}[1, -1]$.

For any $t \in \mathcal{T}$ we define the function $\omega(\,\cdot\,)$ by the equality $\omega(t) \equiv 0$. For all other $t \in [T_k, T_{k+1})$, $k \in \mathbb{N}$, let $q_t := 0$ if $t < \tau_{k,j}$, and $q_t := 1$ in another case, and let $\omega(t) := (-1)^{q_t} b_k(0)$. We define a matrix $C(t)$, $t \geq 0$, by the relations

$$C(t) := U^{-1}(\tau)\Big(\widetilde{A}_1(t)U(\tau) - \frac{\mathrm{d}}{\mathrm{d}t}U(\tau)\Big), \ \ t \geqslant 0, \ \ \tau = \tau(t) := \int\limits_0^t \omega(s)\,ds. \tag{1}$$

Next statement contains the main result of this paper.

Theorem. *For every $G_{\delta\sigma}$ set $M \subset \mathbb{R}$, $0 \notin M$, system* (1_μ) *with the matrix $C(\,\cdot\,)$, given by equality* (1)*, is irregular under Lyapunov for all $\mu \in M$ and is regular for any other $\mu \in \mathbb{R}$.*

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On Positive Periodic Solutions to Parameter-Dependent Second-Order Differential Equations with a Sub-Linear Non-Linearity

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We are interested in the existence and non-existence of a **positive** solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega).} \tag{0.1}$$

Here, $p, h, f \in L([0,\omega])$,

$$h(t) \geq 0 \;\text{ for a.e. }\; t \in [0,\omega], \;\; h(t) \not\equiv 0,$$

$\lambda \in]0,1[$, and a parameter $\mu \in \mathbb{R}$. By a *solution* to problem (0.1), as usually, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere, and verifies periodic conditions.

Definition 0.1. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \;\text{ for a.e. }\; t \in [0,\omega], \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \;\text{ for }\; t \in [0,\omega] \quad \big(\text{resp. }\; u(t) \leq 0 \;\text{ for }\; t \in [0,\omega]\big)$$

is fulfilled.

Definition 0.2. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{0.2}$$

has a positive solution.

For the cases $p \in \mathcal{V}^-(\omega)$, $p \in \mathcal{V}_0(\omega)$, and $p \in \mathcal{V}^+(\omega)$, we provide some results concerning the existence or non-existence of positive solutions to problem (0.1) depending on the choice of a parameter μ.

1 The case $p \in \mathcal{V}^-(\omega)$

Theorem 1.1. *Let $p \in \mathcal{V}^-(\omega)$ and*

$$\int_0^\omega [f(t)]_- \,\mathrm{d}t > \exp\bigg(\int_0^\omega [p(t)]_+ \,\mathrm{d}t\bigg) \int_0^\omega [f(t)]_+ \,\mathrm{d}t.$$

Then there exists $\mu_* \geq 0$ such that

- *for any $\mu > \mu_*$, problem* (0.1) *has a unique positive solution,*
- *for any $\mu \leq \mu_*$, problem* (0.1) *has no positive solution.*

Theorem 1.1 yields immediately the following result.

Theorem 1.2. *Let $p \in \mathcal{V}^-(\omega)$ and*

$$\int_0^\omega [f(t)]_+ \,\mathrm{d}t > \exp\bigg(\int_0^\omega [p(t)]_+ \,\mathrm{d}t\bigg) \int_0^\omega [f(t)]_- \,\mathrm{d}t.$$

Then there exists $\mu^* \leq 0$ such that

- *for any $\mu < \mu^*$, problem* (0.1) *has a unique positive solution,*
- *for any $\mu \geq \mu^*$, problem* (0.1) *has no positive solution.*

2 The case $p \in \mathcal{V}_0(\omega)$

Theorem 2.1. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int_0^\omega f(t)u_0(t) \,\mathrm{d}t < 0,$$

where u_0 is a positive solution to problem (0.2)*. Then there exists $\mu_* \geq 0$ such that*

- *for any $\mu > \mu_*$, problem* (0.1) *has a unique positive solution,*
- *for any $\mu \leq \mu_*$, problem* (0.1) *has no positive solution.*

From Theorem 2.1, we immediately derive the following result.

Theorem 2.2. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int_0^\omega f(t)u_0(t) \,\mathrm{d}t > 0,$$

where u_0 is a positive solution to problem (0.2)*. Then there exists $\mu^* \leq 0$ such that*

- *for any $\mu < \mu^*$, problem* (0.1) *has a unique positive solution,*
- *for any $\mu \geq \mu^*$, problem* (0.1) *has no positive solution.*

3 The case $p \in \mathcal{V}^+(\omega)$

Theorem 3.1. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{3.1}$$

be non-negative. Then there exists $-\infty < \mu_* < 0$ such that

- *for any $\mu > \mu_*$, problem* (0.1) *has a positive solution,*
- *for any $\mu < \mu_*$, problem* (0.1) *has no positive solution.*

Remark 3.1. The assumption about the non-negativity of u in Theorem 3.1 is meaningful. For instance, it follows from Definition 0.1 that the solution u to problem (3.1) is non-negative provided

$$f(t) \geq 0 \ \text{ for a.e. } \ t \in [0, \omega].$$

Moreover, it is known that if

$$\int_0^\omega [f(t)]_+ \, \mathrm{d}t > \Delta(p) \int_0^\omega [f(t)]_- \, \mathrm{d}t,$$

where $\Delta(p)$ is a number depending only on p, then the solution u to problem (3.1) is positive.

Theorem 3.1 yields immediately the following result.

Theorem 3.2. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

be non-positive. Then there exists $0 < \mu^* < +\infty$ such that

- *for any $\mu < \mu^*$, problem* (0.1) *has a positive solution,*
- *for any $\mu > \mu^*$, problem* (0.1) *has no positive solution.*

The last statement complements Theorems 3.1 and 3.2.

Theorem 3.3. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and the solution u to the problem*

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega)$$

change its sign. Then there exist $-\infty < \mu_* < 0$ and $0 < \mu^* < +\infty$ such that

- *for any $\mu \in \,]\mu_*, \mu^*[\,$, problem* (0.1) *has a positive solution,*
- *for any $\mu \in \,]-\infty, \mu_*[\, \cup \,]\mu^*, +\infty[\,$, problem* (0.1) *has no positive solution.*

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Characterizing the Formation of Singularities in a Superlinear Indefinite Mean Curvature Problem

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In this contribution, based on the very recent paper [21], we analyze the quasilinear indefinite Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda a(x) f(u) \quad \text{in } (0,1),\\ u'(0) = u'(1) = 0. \end{cases} \tag{1}$$

Here, $\lambda \in \mathbb{R}$ is regarded as a parameter and

(a_1) the function $a \in L^\infty(0,1)$ satisfies, for some $z \in (0,1)$, $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$, as well as $\int\limits_0^1 a(x)\,dx < 0$;

(f_1) the function $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1[0,+\infty)$ satisfies $f(s) > 0$ and $f'(s) \geq 0$ for all $s > 0$, and there exist four constants, $h > 0$, $k > 0$, $q > 1$ and $p \geq 2$, such that

$$\lim_{s\to+\infty} \frac{f(s)}{s^{q-1}} = qh, \quad \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = pk.$$

Condition (f_1) implies that the potential F of f, defined by $F(s) = \int\limits_0^s f(t)\,dt$, satisfies

$$\lim_{s\to+\infty} \frac{F(s)}{s^q} = h, \quad \lim_{s\to 0^+} \frac{F(s)}{s^p} = k$$

and, thus, F must be superlinear at $+\infty$ and either quadratic or superquadratic at 0. We also introduce the following condition on the weight function a at the nodal point z, which is going to play a pivotal role in the mathematical analysis carried out in [21]

(a_2) $\quad \Big(\int\limits_x^z a(t)\,dt\Big)^{-\frac{1}{2}} \in L^1(0,z)$ and $\Big(\int\limits_x^z a(t)\,dt\Big)^{-\frac{1}{2}} \in L^1(z,1)$.

We use the following notions of a solution.

- A couple (λ, u) is said to be a *regular solution* of (1) if $u \in W^{2,1}(0,1)$ and it satisfies the differential equation a.e. in $(0,1)$, as well as the boundary conditions.

- A couple (λ, u) is said to be a *bounded variation solution* of (1) if $u \in BV(0,1)$ and it satisfies

$$\int_0^1 \frac{D^a u\, D^a \phi}{\sqrt{1+|D^a u|^2}}\,dx + \int_0^1 \frac{D^s u}{|D^s u|}\,D^s\phi = \int_0^1 \lambda a f(u)\phi\,dx$$

  for all $\phi \in BV(0,1)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ (cf. [2]).

- A couple (λ, u) is said to be a *singular solution* of (1) whenever it is a non-regular bounded variation solution; that is, $u \in BV(0,1) \setminus W^{2,1}(0,1)$.

- When the couple (λ, u) solves (1) in any of the previous senses, it is said that (λ, u) is a *positive solution* if, in addition,

$$\lambda > 0, \quad \operatorname{ess\,inf} u > 0.$$

As usual, for any function $v \in BV(0,1)$,

$$Dv = D^a v\,dx + D^s v$$

stands for the Lebesgue decomposition of the Radon measure Dv and $\frac{D^s v}{|D^s v|}$ denotes the density function of the measure $D^s v$ with respect to its total variation $|D^s v|$ (see [1]). By [23, Prop. 3.6], any positive singular solution, (λ, u), of (1) actually satisfies

$$\begin{aligned} u\big|_{[0,z)} &\in W^{2,1}_{\rm loc}[0,z) \cap W^{1,1}(0,z) \text{ and is concave,}\\ u\big|_{(z,1]} &\in W^{2,1}_{\rm loc}(z,1] \cap W^{1,1}(z,1) \text{ and is convex;} \end{aligned} \tag{2}$$

moreover, $u'(x) < 0$ for every $x \in (0,1) \setminus \{z\}$, $u'(0) = u'(1) = 0$ and

$$u'(z^-) = u'(z^+) = -\infty,$$

where $u'(z^-)$ and $u'(z^+)$ are the left and the right Dini derivatives of u at z. The same argument used in [23, Lem. 2.1] shows that $\lambda > 0$ is necessary for the existence of positive non-constant, either regular or singular, solutions.

Problem (1) is a one-dimensional prototype model of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) & \text{in } \Omega,\\ -\dfrac{\nabla u\cdot\nu}{\sqrt{1+|\nabla u|^2}} = \sigma & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. Problem (3) plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [3, 9, 12–15, 19, 25, 26], capillarity phenomena for incompressible fluids [6, 10, 11, 16, 17], and reaction-diffusion processes where the flux features saturation at high regimes [5, 18, 24].

The model (1) has been recently investigated by the authors in [22, 23] and [20]. In [22] the existence of bounded variation solutions was analyzed by using variational methods and in [23] the existence of regular solutions was dealt with by means of classical phase plane and bifurcation techniques. The main result of [20] established the existence of a component of bounded variation solutions bifurcating from the trivial state $(\lambda, 0)$ in the special, but significant, case where $p = 2$. According to the results of these papers, it is already known that, under conditions (a_1) and (f_1), problem (1) cannot admit positive solutions if $\lambda < 0$ and that it possesses at least one positive bounded variation solution for sufficiently small $\lambda > 0$.

Quite strikingly, whether or not these bounded variation solutions are singular depends on whether or not condition (a_2) holds true: this is the main result of [21] which can be stated as follows.

Theorem 1. *Assume* (a_1) *and* (f_1). *Then, the following conclusions hold for sufficiently small* $\lambda > 0$*:*

(i) *any positive solution of* (1) *is singular if* (a_2) *holds;*

(ii) *any positive solution of* (1) *is regular if* (a_2) *fails.*

In other words, condition (a_2) completely characterizes, under (a_1) and (f_1), the development of singularities by the positive solutions of (1) for sufficiently small $\lambda > 0$.

By having a glance at condition (a_2) it becomes apparent that it fails whenever the function a is differentiable at the nodal point z, whereas a very simple example where (a_2) holds occurs when the function a is discontinuous at z, like, for instance, in the special case when a is assumed to be a positive constant, $A > 0$, in $[z - \eta_1, z)$ and a negative constant, $-B < 0$, in $(z, z + \eta_2]$, for some $\eta_1, \eta_2 > 0$. The huge contrast on the nature of the positive solutions of the problem with respect to the integrability properties of the function a near the node z can also be realized by considering any weight function a satisfying the requirements of (a_1) except for the fact that $a = 0$ in $[z - \eta, z + \eta]$ for some $\eta > 0$. In such case, thanks to the convexity and concavity properties of the positive bounded variation solutions of (1) guaranteed by [23, Prop. 3.6], any positive solution u must be linear in the interval $[z - \eta, z + \eta]$ and hence, due to (2), it cannot develop singularities.

As a consequence of Theorem 1, when $p = 2$, the global structure of the component of the positive solutions of (1), $\mathscr{C}_+$, whose existence is guaranteed by the main theorem of [20], drastically changes according to whether or not the condition (a_2) holds as illustrated in Figure 1, where λ_0 stands for the principal positive eigenvalue of the linear weighted problem

$$\begin{cases} -\varphi'' = \lambda a(x)\varphi & \text{in } (0,1), \\ \varphi'(0) = \varphi'(0) = 0. \end{cases}$$

The non-existence of positive regular solutions of (1) in the very special cases when $p = 2$ and the weight a is constant in $[0, z)$ and in $(z, 1]$ has been recently established in Section 8 of [23] by using some classical, but sophisticated, phase portrait techniques. This induced the authors to presume that an analogous non-existence result should also be valid for general weight functions a, without imposing the integrability condition (a_2). So, they formulated [23, Th. 7.1]. Theorem 1 in particular shows that [23, Th. 7.1] has to be complemented with condition (a_2).

Similarly as for $p = 2$, also in the case $p > 2$ the global structure of the set of positive solutions of (1), $\mathscr{C}_+$, whose existence is now guaranteed by [22, Th. 1.1] and [23, Th. 10.1], changes for sufficiently small $\lambda > 0$ according to whether or not condition (a_2) holds, as illustrated by Figure 2.

Our proof of Theorem 1 is based upon the characterization of the exact limiting profiles of the positive solutions of (1), both regular and singular, as the parameter λ approximates zero. These profiles are provided by the next theorem, regardless their particular nature.

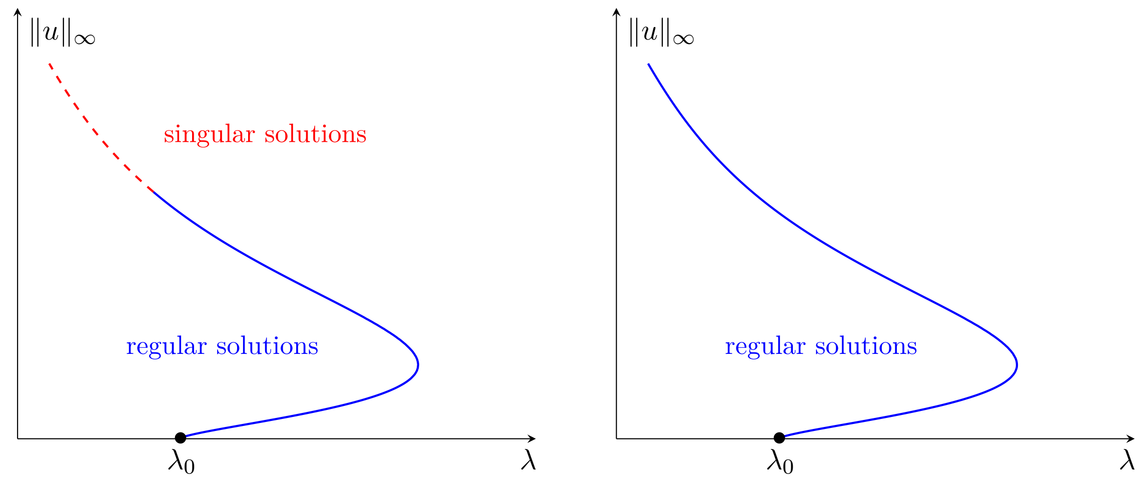

Figure 1. Global components emanating from the positive principal eigenvalue λ_0 in case $p=2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

Theorem 2. *Assume* (a_1) *and* (f_1), *and let* $((\lambda_n,u_n))_n$ *be an arbitrary sequence of positive solutions of* (1) *with* $\lim\limits_{n\to\infty}\lambda_n=0$. *Then, for sufficiently small* $\eta>0$, *the following assertions hold:*

$$\lim_{n\to+\infty}\frac{u_n(x)}{u_n(0)}=1 \ \ \textit{uniformly in} \ \ x\in[0,z-\eta],$$

$$\lim_{n\to+\infty}\frac{u_n(x)}{u_n(0)}=\left(\frac{\int\limits_0^z a(t)\,dt}{-\int\limits_z^1 a(t)\,dt}\right)^{\frac{1}{q-1}} \ \ \textit{uniformly in} \ \ x\in[z+\eta,1],$$

$$\lim_{n\to+\infty}(\lambda_n f(u_n(x)))=\frac{1}{\int\limits_0^z a(t)\,dt} \ \ \textit{uniformly in} \ \ x\in[0,z-\eta],$$

$$\lim_{n\to+\infty}(\lambda_n f(u_n(x)))=\frac{1}{-\int\limits_z^1 a(t)\,dt} \ \ \textit{uniformly in} \ \ x\in[z+\eta,1],$$

$$\lim_{n\to+\infty}u_n'(x)=\frac{-\int\limits_0^x a(t)\,dt}{\sqrt{\Big(\int\limits_0^z a(t)\,dt\Big)^2-\Big(\int\limits_0^x a(t)\,dt\Big)^2}} \ \ \textit{uniformly in} \ \ x\in[0,z-\eta],$$

and

$$\lim_{n\to+\infty}u_n'(x)=\frac{\int\limits_x^1 a(t)\,dt}{\sqrt{\Big(\int\limits_z^1 a(t)\,dt\Big)^2-\Big(\int\limits_x^1 a(t)\,dt\Big)^2}} \ \ \textit{uniformly in} \ \ x\in[z+\eta,1].$$

Note that condition (a_2) is equivalent to requiring the integrability in both intervals, $(0,z)$ and $(z,1)$, of the asymptotic profile of the derivatives of the positive solutions of (1) as $\lambda\to 0^+$, which

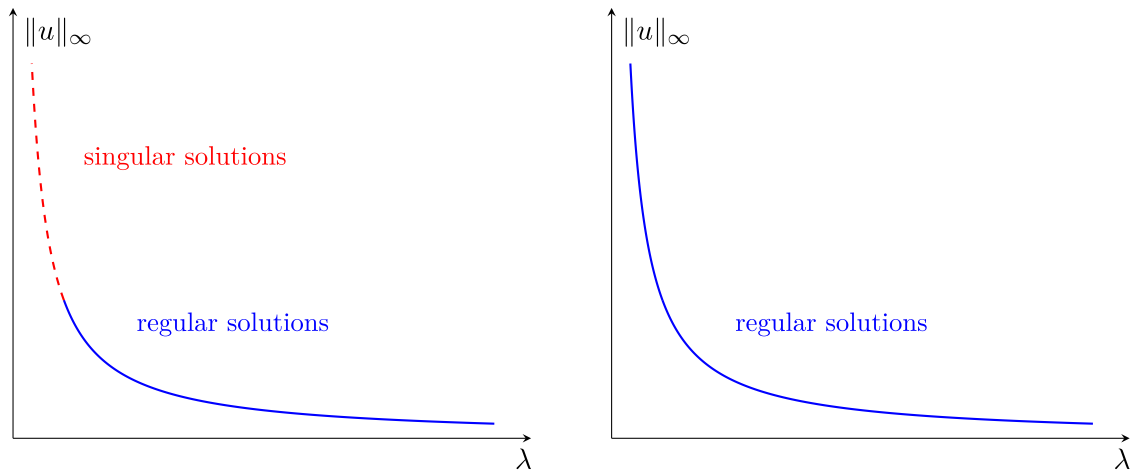

Figure 2. Global bifurcation diagrams in case $p > 2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

is equivalent to impose that the "limiting derivative"

$$u'_\omega(x) = \begin{cases} \dfrac{-\int\limits_0^x a(t)\,dt}{\sqrt{\Big(\int\limits_0^z a(t)\,dt\Big)^2 - \Big(\int\limits_0^x a(t)\,dt\Big)^2}} & \text{for } \ x \in [0,z), \\[2ex] \dfrac{\int\limits_x^1 a(t)\,dt}{\sqrt{\Big(\int\limits_z^1 a(t)\,dt\Big)^2 - \Big(\int\limits_x^1 a(t)\,dt\Big)^2}} & \text{for } \ x \in (z,1], \end{cases}$$

belongs to both $L^1(0,z)$ and $L^1(z,1)$.

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On Some Fine Properties of Supercritical Sigma-Perturbations

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Consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A such that $\|A(t)\| \leq M < +\infty$ for all $t \geq 0$. We denote the Cauchy matrix of (1) by X_A and the highest Lyapunov exponent of (1) by $\lambda_n(A)$. Together with system (1) consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with piecewise continuous and bounded perturbation matrix Q such that

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \ \ t \geq 0. \tag{3}$$

Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

Let $\mathfrak{M}_\sigma(A)$ be the set of all perturbations Q satisfying condition (3) and having the appropriate dimensions. Any $Q \in \mathfrak{M}_\sigma$ is said to be a sigma-perturbation and the number $\nabla_\sigma(A) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}_\sigma(A)\}$ is called [7], [10, p. 225], [9, p. 214] the highest sigma-exponent or the Izobov exponent of system (1). It was proved in [7] that the Izobov exponent can be evaluated by means of the following algorithm:

$$\nabla_\sigma(A) = \varlimsup_{m\to\infty} \frac{\xi_m(\sigma)}{m}\,, \tag{4}$$
$$\xi_m(\sigma) = \max_{k<m} \big(\ln\|X_A(m,k)\| + \xi_k(\sigma) - \sigma k\big), \ \ \xi_1 = 0, \ \ k \in \mathbb{N}.$$

According to [1,11], there exists a unique critical value $\sigma_0(A) \geq 0$ such that $\nabla_\sigma(A) = \lambda_n(A)$ for all $\sigma \geq \sigma_0(A)$ and $\nabla_\sigma(A) > \lambda_n(A)$ when $0 < \sigma < \sigma_0(A)$. It is well known that $\nabla_\sigma(A) = \lambda_n(A)$ for all $\sigma > 2M$ and, therefore, $\sigma_0(A) \leq 2M$. Using the Lyapunov $\sigma_{\rm L}(A)$, Grobman $\sigma_{\rm G}(A)$ or Perron $\sigma_{\rm P}(A)$ irregularity coefficients [4, pp. 67, 73], [8, pp. 77, 81] one can obtain some more accurate estimates for $\sigma_0(A)$. Indeed, the inequalities $\sigma_0(A) \leq \sigma_{\rm L}(A)$ and $\sigma_0(A) \leq \sigma_{\rm G}(A)$ were proved in [3] and [5]. It was also proved that the inequality $\sigma_0(A) \leq \sigma_{\rm P}(A)$ holds for $n = 2$, see [6], and is not valid for $n > 2$, see [12,15]. These relations are combined in [15], where the irregularity quantity $\sigma_\lambda(A)$ is constructed in such a way that $\sigma_{\rm G}(A) \geq \sigma_\lambda(A) \geq \sigma_0(A)$ for all $n \in \mathbb{N}$ and $\sigma_\lambda(A) = \sigma_{\rm P}(A)$ for $n = 2$.

In [13] we give an explicit formula for evaluation of $\sigma_0(A)$ from the Cauchy matrix X_A of the original system. To formulate this result we need some notation.

Let $\mathcal{D}(m)$ be the set of all nonempty $d \subset \{1,\dots,m-1\} \subset \mathbb{N}$. Further we assume that for each $d \in \mathcal{D}(m)$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \cdots < d_s$ and $d = \{d_1, d_2, \dots, d_s\}$, where $s = |d|$ is the number of elements of the set d. We also put $\|d\| := d_1 + \cdots + d_s$ for $d \in \mathcal{D}(m)$ and $\|d\| := 0$ for $d = \varnothing$. In addition, for the sake of convenience we assume that $d_0 = 0$ and $d_{s+1} = m$ for each $d \in \mathcal{D}_0(m) := \mathcal{D}(m) \cup \{\varnothing\}$. Note that we do not include these additional elements in the set d. Under the above assumptions, let us define the quantity $\Xi(m,d)$ as

$$\Xi(m,d) := \sum_{i=0}^{s} \ln \|X_A(d_{i+1},d_i)\|,$$

where $m \in \mathbb{N}$, $d \in \mathcal{D}(m)$ and $s := |d|$. From [2,14] we can assert that

$$\xi_m(\sigma) = \max_{d\in\mathcal{D}_0(m)} \big(\Xi(m,d) - \sigma\|d\|\big). \tag{5}$$

Theorem 1 ([13])**.** *The equality*

$$\sigma_0(A) = \varlimsup_{m\to\infty} \max_{d\in\mathcal{D}(m)} \|d\|^{-1}(\Xi(m,d) - m\lambda_n(A)) \tag{6}$$

holds.

Theorem 2 ([13])**.** *The estimate*

$$\sigma_0(A) \geq \sigma^+ := \varlimsup_{m\to\infty} \max_{k<m} k^{-1}\big(\ln\|X_A(m,k)\| + \ln\|X_A(k,0)\| - m\lambda_n(A)\big) \tag{7}$$

is valid. If the limit $\lim\limits_{m\to\infty} m^{-1}\ln\|X_A(m,0)\|$ *exists, then* $\sigma_0(A) = \sigma^+$.

These theorems are obtained by direct inversion of (4) and (5) using some standard tools of convex analysis.

Since $\sigma_0(A)$ is said to be a critical value, we can say that all sigma-perturbations with $\sigma > \sigma_0(A)$ are supercritical. In order to investigate some fine properties of such perturbations we should modify the above expressions. It seems to be a natural idea to replace $m\lambda_n(A)$ by $\ln\|X_A(m,0)\|$ in (6) or (7). In this way we put

$$\sigma^{\#}(A) = \varlimsup_{m\to\infty} \max_{d\in\mathcal{D}(m)} \|d\|^{-1}\big(\Xi(m,d) - \ln\|X_A(m,0)\|\big).$$

Evidently, $\sigma^{\#}(A) \geq \sigma_0(A)$.

Let X_{A+Q} be the Cauchy matrix of system (2). Using the estimates for the norm of X_{A+Q} obtained in [14] we can prove the following statement.

Theorem 3. *If* $\sigma > \sigma^{\#}(A)$, *then* $\|X_{A+Q}(t,0)\| \leq K\|X_A(t,0)\|$ *with some* $K > 0$ *for all* $t > 0$. *If* $\sigma < \sigma^{\#}(A)$, *then* $\|X_{A+Q}(t,0)\|\|X_A(t,0)\|^{-1}$ *is unbounded as* $t \to +\infty$.

It should be noted that to reveal the meaning of

$$\sigma^{\Delta} := \varlimsup_{m\to\infty} \max_{k<m} k^{-1}\big(\ln\|X_A(m,k)\| + \ln\|X_A(k,0)\| - \ln\|X_A(m,0)\|\big)$$

still remains an open problem.

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A Class of Continuous-Discrete Functional Differential Equations with the Cauchy Operator Constructed Explicitly

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1 Introduction

Here we follow the previous works [2–4] and consider the linear continuous-discrete functional differential system

$$\delta y = \mathcal{T} y + r, \tag{1.1}$$

where $y = col(x, z), r = col(f, g), x : [0, T] \to R^n,\ z : \{0, t_1, \dots, t_\mu\} \to R^\nu,\ \delta y = col(\dot{x}, z),\ \mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix}$, and $\mathcal{T}_{11} : AC^n \to L^n$, $\mathcal{T}_{12} : FD^\nu \to L^n$, $\mathcal{T}_{21} : AC^n \to FD^\nu$, $\mathcal{T}_{22} : FD^\nu \to FD^\nu$ are linear Volterra operators. Here L^n is the space of summable functions $f : [0, T] \to R^n$, AC^n is the space of absolutely continuous functions $x : [0, T] \to R^n$, the space FD^ν is defined by the given set $J = \{0, t_1, \dots, t_\mu\}, 0 = t_0 < t_1 < \cdots < t_\mu = T$, as the space of functions $z : J \to R^\nu$. The spaces L^n, AC^n and FD^ν are assumed to be equipped with natural norms.

It should be noted that the system (1.1) can be considered as a concrete realization of the so-called Abstract Functional Differential Equation, the theory of which is thoroughly treated in [1]. The systems of the kind (1.1) arise in particular as dynamic models in Mathematical Economics and cover many kinds of systems with aftereffect. Representation of solutions to some classes of dynamic models close to (1.1) and discussion of actual applied problems can be found in [10]. The questions of stability to functional differential systems with continuous and discrete times are studied in [13].

The central point of the consideration is the representation of solutions to (1.1). The structure and some principal properties of the Cauchy operator are described in [8] with the use of the general representation to the operators $\mathcal{T}_{ij},\ i, j = 1, 2$. The main aim of this paper is to give an explicit representation for the components of the Cauchy operator in a special case.

2 The Cauchy operator

Let V be the integration operator: $(Vu)(t) = \int\limits_0^t u(s)\,ds$, and $K = \mathcal{T}_{11} V$ be an integral operator with the kernel $K(t, s) = (k_{ij}(t, s))$ that satisfies the condition $\mathcal{K}$: for all the elements k_{ij}, there exists a common summable majorant $\kappa(\cdot)$, $|k_{ij}(t, s)| \le \kappa(t), t \in [0, T]$.

Let us recall some general results [2] for the case that the condition $\mathcal{K}$ is fulfilled.

The general solution of (1.1) has the representation

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{X} \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} + \mathcal{C} \begin{pmatrix} ccf \\ g \end{pmatrix},$$

where $\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix}$ is the fundamental operator (fundamental matrix), $\mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix}$ is the Cauchy operator.

Denote by C_1 and $X(t)$ the Cauchy operator and the fundamental matrix to the equations $\dot{x} = \mathcal{T}_{11}x$, and denote by C_2 and $Z(t_i)$ the Cauchy operator and the fundamental matrix to the equation $z = \mathcal{T}_{22}z$.

Define the operators $\mathcal{H}_{ij}$, $i,j = 1,2$ by the equalities

$$\begin{gathered}\mathcal{H}_{11} = (I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}, \quad \mathcal{H}_{12} = -(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{12})^{-1}C_1\mathcal{T}_{21}, \\ \mathcal{H}_{21} = C_2\mathcal{T}_{21}(I - C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1}, \quad \mathcal{H}_{22} = (I - C_2\mathcal{T}_{21}C_1\mathcal{T}_{12})^{-1},\end{gathered}$$

where I is the identity operator.

Theorem 2.1 ([9])**.** *The Cauchy operator $\mathcal{C} = (\mathcal{C}_{ij})$ of* (1.1) *is defined by the equalities*

$$\mathcal{C}_{ij} = \mathcal{H}_{ij}C_j, \ \ i,j = 1,2.$$

It should be noted that C_2 can be constructed in the explicit form. From Theorem 2.1 it follows that the component C_1 is of principal interest and requires the development of efficient algorithms to approximate construction of it. Some of those are described in [7].

In what follows we shall construct the Cauchy operator for the following continuous-discrete functional differential system

$$\dot{x}(t) = \sum_{i:\, t_i<t} A_i(t)x(t_i) + \sum_{i:\, t_i<t} B_i(t)z(t_i) + f(t), \ \ t \in [0,T], \tag{2.1}$$

$$z(t_i) = \sum_{j<i} D_j x(t_j) + \sum_{j<i} H_j z(t_j) + g(t_i), \ \ i = 1,\dots,\mu \tag{2.2}$$

with summable $(n \times n)$-matrices $A_i(t)$, $(n \times \nu)$-matrices $B_i(t)$ and constant $(\nu \times n)$-matrices D_j, $(\nu \times \nu)$-matrices H_j.

Let us define the operator $\Theta : AC^n \to L^n$ by the equality

$$(\Theta\, x)(t) = \sum_{i:\, t_i<t} A_i(t)x(t_i) + \sum_{i:\, t_i<t} B_i(t)\Big[\sum_{j=1}^{i} C_2(i,j)\sum_{k<j} D_k\, x(t_k)\Big].$$

After some transformations this operator can be represented in the form

$$(\Theta\, x)(t) = \sum_{i:\, t_i<t} \mathcal{A}_i(t)x(t_i), \tag{2.3}$$

where the matrices $\mathcal{A}_i(t)$ are calculated by $A_i(t)$, $B_i(t)$, $C_2(i,j)$, D_i.

Denote by $C(t,s)$ the Cauchy matrix [5] to the equation $\dot{x} = \Theta x$.

As is shown in [7, Theorem 1, Remark 2], $C(t,s)$ can be constructed explicitly. Let us recall the main relationships from [7]. Let $\eta_i(t)$, $i = 1,\dots,\mu-1$, be the characteristic function of the set $[t_{i-1},t_i)$, and $\eta_\mu(t)$ denotes the characteristic function of the segment $[t_{\mu-1},t_\mu]$. Define the kernel of the integral operator $(Kz)(t) = \int\limits_0^t K(t,s)z(s)\,ds$ by the equality

$$K(t,s) = \sum_{i=1}^{\mu}\sum_{j=1}^{i} \eta_i(t)P_i(t)Q_{ij}(s)\eta_j(s), \tag{2.4}$$

where $P_i(t)$ and $Q_{ij}(s)$ are $(n\times n)$-matrices, $P_1(t)=0$, $Q_{ij}(s)=0$, $j\geq i$, elements of P_i are summable on $[0,T]$, elements of Q_{ij} are measurable and essentially bounded on $[0,T]$. Next define the matrices B_{ki} by the equalities

$$B_{ki}=\int\limits_0^T\sum_{j=1}^k Q_{kj}(t)\eta_j(t)\eta_i(t)P_i(t)\,dt.$$

Notice that by definition the block matrix $G=\{G_{ki}\}_{k,i=1,\dots,\mu}$, $G_{kk}=E_n$, $k=1,\dots,\mu$, where E_n is the identity $(n\times n)$-matrix, $G_{ki}=-B_{ki}$, is a lower triangle matrix with E_n as the diagonal blocks. Finally denote by F_{ki} the block elements of the inverse G^{-1}. By Theorem 1 of [7] we have the explicit representation of the resolvent kernel $R(t,s)$ for the kernel $K(t,s)$ defined by (2.4):

$$R(t,s)=\sum_{i=1}^{\mu}\sum_{k=1}^{\mu}\sum_{j=1}^{k}\eta_i(t)P_i(t)F_{ik}Q_{kj}(s)\eta_j(s),$$

and

$$C(t,s)=E_n+\int\limits_s^t R(\tau,s)\,d\tau.$$

It remains to note that, for the operator Θ (2.3), we have

$$(\Theta Vu)(t)=\int\limits_0^t\sum_{i\colon t_i<t}\mathcal{A}_i(t)\eta_i(s)u(s)\,ds,$$

and this is the integral operator with the kernel of the kind (2.4). Now we are ready to give the representations of the fundamental matrix $\mathcal{X}$ and the Cauchy operator $\mathcal{C}$ for the system (2.1), (2.2) in terms of $X(t)$, $Z(t_i)$, $C(t,s)$ and $C_2(i,j)$.

Theorem 2.2. *The representation of the components to the fundamental matrix and the Cauchy operator of* (2.1), (2.2) *is defined by the equalities*

$$\mathcal{X}_{11}(t)=X(t),\quad \mathcal{X}_{12}(t)=\int\limits_0^t C(t,s)\Big[\sum_{i\colon t_i<t}B_i(t)Z(t_i)\Big]\,ds,$$

$$\mathcal{X}_{21}(t_i)=\sum_{j=1}^{i}C_2(i,j)\Big[\sum_{k<j}D_k\mathcal{X}_{11}(t_k)\Big],\quad \mathcal{X}_{22}(t_i)=Z(t_i)+\sum_{j=1}^{i}C_2(i,j)\Big[\sum_{k<j}D_k\mathcal{X}_{11}(t_k)\Big],$$

$$(\mathcal{C}_{11}f)(t)=\int\limits_0^t C(t,s)f(s)\,ds,\quad (\mathcal{C}_{12}g)(t)=\int\limits_0^t C(t,s)\sum_{i\colon t_i<s}B_i(s)\Big[\sum_{j=1}^{i}C_2(i,j)g(t_j)\Big]\,ds,$$

$$(\mathcal{C}_{21}f)(t_i)=\sum_{j=1}^{i}C_2(i,j)\Big[\sum_{k<j}D_k(\mathcal{C}_{11}f)(t_k)\Big],\quad (\mathcal{C}_{22}g)(t_i)=\sum_{j=1}^{i}C_2(i,j)\Big[\sum_{k<j}D_k(\mathcal{C}_{12}g)(t_k)+g(t_j)\Big].$$

The systems (2.1), (2.2) are actively studied as models of some dynamic economic processes [12]. Furthermore, they can be used as approximations of more general systems (1.1) which opens the way to obtaining external estimates of attainability sets for control problems [6, 11].

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Disconjugacy and Solvability of Dirichlet BVP for the Fourth Order Ordinary Differential Equations

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Consider on the interval $I=[a,b]$ the fourth order homogeneous linear ordinary differential equations

$$u^{(4)}(t)=p(t)u(t)-\mu q(t)u(t), \tag{0.1}$$

$$u^{(4)}(t)=p(t)u(t), \tag{0.2}$$

and the nonlinear equation

$$u^{(4)}(t)=p(t)u(t)+f(t,u(t))+h(t), \tag{0.3}$$

under the boundary conditions

$$u^{(i)}(a)=0, \quad u^{(i)}(b)=0 \ \ (i=0,1), \tag{0.4}$$

where $\mu\in R$, $h\in L(I,R)$, $p,q\in L(I,R_0^+)$, and $f\in K(I\times R,R)$. The study of the fourth order boundary value problems has increased recently, among them because they appear as a model equations for a large class of higher order parabolic equations arising, for instance, in statistical mechanics, phase field models, hydrodynamics, suspension bridges models, etc.

In [6] (see Lemma 4.2) it has been shown that the disconjugacy character of equation (0.1) implies the nonnegativity of Greens's function of problem $(0.1),(0.4)$. However, as we can see in [3], there are coefficients of (0.1), for which Green's function has constant sign but equation (0.1) is not disconjugate on I. For these reasons, we study disconjugacy of equation (0.1) on the interval I in connection with parameter μ, under the assumption that problem $(0.2),(0.4)$ has constant sign nonzero solution (see Definition 0.2). Also we find the necessary and sufficient conditions of nonnegativity of Green's function of problem $(0.1),(0.4)$ when $p\in D(I)$, and on the basis of these results we prove the sufficient conditions of solvability and unique solvability of the nonlinear problem $(0.3),(0.4)$.

The following notations are used throughout the paper.

- $R=]-\infty,+\infty[$, $R^+=]0,+\infty[$, $R_0^+=[0,+\infty[$;
- $C(I;R)$ is the Banach space of continuous functions $u:I\to R$ with the norm $\|u\|_C=\max\{|u(t)|:\ t\in I\}$;
- $\widetilde{C}^3(I;R)$ is the set of functions $u:I\to R$ which are absolutely continuous together with their third derivatives;

- $L(I;R)$ is the Banach space of Lebesgue integrable functions $p: I \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$;
- $K(I \times R;R)$ is the set of functions $f: I \times R \to R$ satisfying the Carathéodory conditions.

By a solution of equation (0.3) we understand a function $u \in \widetilde{C}^3(I,R)$ which satisfies equation (0.3) a.e. on I.

Definition 0.1. Equation (0.1) is said to be disconjugate on I if every nontrivial solution u has less than four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 0.2. We say that $p \in D(I)$ if $p \in L(I;R_0^+)$, and problem $(0.2),(0.4)$ has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,. \tag{0.5}$$

If we consider the equation

$$u^{(4)}(t) = \lambda p(t)u(t), \tag{0.6}$$

the set $D(I)$ can be interpreted as a set of the functions $p \in L(I,R_0^+)$ for which $\lambda = 1$ is the first eigenvalue of problem $(0.6),(0.4)$.

1 Disconjugacy of equation (0.1)

Theorem 1.1. *Let $p \in D(I)$, $q \in L(I,R_0^+)$, $q \not\equiv 0$, and*

$$\mu_1 = \sup\big\{\mu: \ \ p(t) - \mu q(t) \ge 0 \ \text{ a.e. on } I\big\} > 0. \tag{1.1}$$

Then for an arbitrary $\mu \in \,]0,\mu_1]$ equation (0.1) is disconjugate on I.

Remark 1.1. Notice that condition (1.1) holds iff

$$\operatorname{mes}\big\{t \in I: \ p(t) = 0, \ q(t) \neq 0\big\} = 0.$$

Corollary 1.1. *Let $p_0 \in D(I)$, $p \in L(I,R)$, $p \not\equiv p_0$, and*

$$0 \le p(t) \le p_0(t) \ \text{ a.e. on } I. \tag{1.2}$$

Equation (0.2) is disconjugate on I.

From the last Corollary it immediately follows

Corollary 1.2. *Let $\lambda_1 \in R^+$ be such that $\lambda_1 p \in D(I)$. Then equation (0.6) is disconjugate on I for an arbitrary $\lambda \in [0,\lambda_1[\,$.*

Corollary 1.2 for $p \equiv 1$ is proved in [7] (see Theorem 3.1) and is optimal.

Remark 1.2 ([6, Lemma 4.2])**.** If equation (0.1) is disconjugate on I, then problem $(0.1),(0.4)$ has only the trivial solution and its Green's function is nonnegative on $I \times I$.

2 Nonnegativity of Green's function of problem (0.1), (0.4)

The disconjugacy is only a sufficient condition in order to ensure the constant sign of Green's function of problem (0.1), (0.4). Now we give the theorem where necessary and sufficient conditions of nonnegativity of Green's function of problem (0.1), (0.4) are given when $p \in D(I)$, and $q \equiv 1$. Consider for this the boundary conditions

$$u(a) = \cdots = u^{(k-1)}(a) = 0, \quad u(b) = \cdots = u^{(3-k)}(b) = 0, \tag{2.1$_k$}$$

and let

$$\mu_2 = \min\{\mu_1', \mu_3'\},$$

where μ_k' $(k = 1, 3)$ are the least positive eigenvalues of problem (0.1), (2.1$_k$) (The existence of μ_1' and μ_3' for $q \equiv 1$ follows from the prove of Theorem 2.1). Then the next theorem is true.

Theorem 2.1. *Let $p \in D(I) \cap C(I, R^+)$, and $q \equiv 1$. Then problem (0.1), (0.4) has only the zero solution and its Green's function is nonnegative on $I \times I$ if and only if $\mu \in \,]0, \mu_2]$.*

3 Nonlinear problem

Theorem 3.1. *Let problem (0.2), (0.4) be uniquely solvable, its Green's function be nonnegative on $I \times I$, and the condition*

$$f(t,x) \operatorname{sign} x \le \delta(t,x) \ \text{ for } \ |x| > r, \ \ t \in I, \tag{3.1}$$

hold, where $r \in R^+$, $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument, and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int\limits_a^b \delta(s, \rho)\, ds = 0. \tag{3.2}$$

Then problem (0.3), (0.4) has at least one solution.

From the last theorem by Remark 1.2 and Theorems 1.1, 2.1 it immediately follow.

Corollary 3.1. *Let $p \in D(I)$, $q \in L(I, R_0^+)$, $q \not\equiv 0$, and condition (1.1) hold. Let, moreover, conditions (3.1) and (3.2) be fulfilled, where $r \in R^+$, and $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument. Then the equation*

$$u^{(4)}(t) = (p(t) - \mu q(t))u(t) + f(t, u(t)) + h(t) \tag{3.3}$$

under the boundary conditions (0.4) has at least one solution for an arbitrary $\mu \in \,]0, \mu_1]$.

Corollary 3.2. *Let $p \in D(I) \cap C(I, R^+)$, $q \equiv 1$, and μ_2 be the constant defined in Theorem 2.1. Let, moreover, conditions (3.1) and (3.2) be fulfilled, where $r \in R^+$, and $\delta \in K(I \times R, R_+)$ is nondecreasing in the second argument. Then problem (3.3), (0.4) has at least one solution for an arbitrary $\mu \in \,]0, \mu_2]$.*

Theorem 3.2. *Let $p_0 \in D(I)$, $p \in L(I, R)$, $p \not\equiv p_0$,*

$$0 \le p(t) \le p_0(t) \ \text{ for } \ t \in I, \tag{3.4}$$

and the condition

$$-p(t)|x_1 - x_2| \le (f(t, x_1) - f(t, x_2)) \operatorname{sign}(x_1 - x_2) \le 0 \tag{3.5}$$

hold on $I \times R$. Then problem (0.3), (0.4) is uniquely solvable.

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The Dirichlet Problem for Singular Two-Dimensional Linear Differential Systems

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We consider the two-dimensional linear differential system

$$u_i' = p_i(t)u_{3-i} + q_i(t) \;\; (i = 1, 2) \tag{1}$$

with the boundary conditions

$$u_1(a+) = 0, \;\; u_1(b-) = 0, \tag{2}$$

where p_1 and $q_1 :]a, b[\to \mathbb{R}$ are Lebesgue integrable functions, while the functions p_2 and $q_2 :]a, b[\to \mathbb{R}$ are Lebesgue integrable on every closed interval contained in $]a, b[$.

We are mainly interested in the case where the functions p_2 and q_2 have nonintegrable singularities at the points a and b, i.e. the case, where

$$\int\limits_a^b \big(|p_2(t)| + |q_2(t)|\big)\,dt = +\infty.$$

System (1) is singular in that sense.

We have proved the theorem on the Fredholmity of problem (1), (2), and based on this theorem we have established unimprovable in a certain sense conditions guaranteeing the unique solvability of the above-mentioned problem. They are generalizations of some results by T. Kiguradze [1], concerning the unique solvability of the Dirichlet problem for singular second order linear differential equations.

We use the following notation.

$$[x]_+ = \frac{|x| + x}{2}\,, \quad [x]_- = \frac{|x| - x}{2}\,;$$

$u(t_0+)$ and $u(t_0-)$ are the right and the left limits, respectively, of the function u at the point t_0;

$L([a, b])$ is the space of Lebesgue integrable on $[a, b]$ real functions;

$L_{loc}(]a, b[)$ is the space of real functions which are Lebesgue integrable on every closed interval contained in $]a, b[$;

If $p \in L([a, b])$, then

$$I_{a,b}(p)(t) = \int\limits_a^t p(s)\,ds \int\limits_t^b p(s)\,ds \;\text{ for }\; a \le t \le b.$$

A vector-function $(u_1, u_2) :]a, b[\to \mathbb{R}^2$ is said to be **a solution of system** (1) if its components are absolutely continuous on every closed interval contained in $]a, b[$ and satisfy system (1) almost everywhere on $]a, b[$.

A solution of system (1) satisfying the boundary conditions (2) is said to be **a solution of problem** (1)**,** (2).

Everywhere below it is assumed that

$$
\begin{gathered}
p_1 \in L([a,b]), \quad q_1 \in L([a,b]), \\
p_2 \in L_{loc}(]a,b[), \quad q_2 \in L_{loc}(]a,b[).
\end{gathered}
$$

Along with system (1) we consider the corresponding homogeneous system

$$
u_i' = p_i(t)u_{3-i} \quad (i=1,2). \tag{1_0}
$$

Theorem 1. *Let the functions p_1 and p_2 satisfy the conditions*

$$
p_1(t) \geq 0 \;\; \text{for} \;\; a<t<b, \quad \delta = \int\limits_a^b p_1(t)\,dt > 0, \tag{3}
$$

$$
\int\limits_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt < +\infty, \tag{4}
$$

and let the functions q_1 and q_2 satisfy the conditions

$$
\int\limits_a^b I_{a,b}(p_1)(t)\big(I_{a,b}(|q_1|)(t)[p_2(t)]_+ + |q_2(t)|\big)\,dt < +\infty. \tag{5}
$$

If, moreover, the homogeneous problem $(1_0), (2)$ has only the trivial solution, then problem $(1), (2)$ has one and only one solution.

Remark 1. If

$$
\begin{gathered}
\limsup_{t\to a+} \frac{p_1(t)}{(t-a)^{\alpha_0}} < +\infty, \quad \limsup_{t\to b-} \frac{p_1(t)}{(b-t)^{\beta_0}} < +\infty, \\
\limsup_{t\to a+} \frac{|q_1(t)|}{(t-a)^{\alpha_1}} < +\infty, \quad \limsup_{t\to b-} \frac{|q_1(t)|}{(b-t)^{\beta_1}} < +\infty,
\end{gathered}
$$

where $\alpha_i > -1$, $\beta_i > -1$ $(i=0,1)$, then for conditions (4) and (5) to be satisfied it is sufficient that the conditions

$$
\int\limits_a^b (t-a)^{\alpha_0+1}(b-t)^{\beta_0+1}[p_2(t)]_-\,dt < +\infty,
$$

$$
\int\limits_a^b \Big[(t-a)^{\alpha_0+\alpha_1+2}(b-t)^{\beta_0+\beta_1+2}\big[p_2(t)\big]_+ + (t-a)^{\alpha_0+1}(b-t)^{\beta_0+1}|q_2(t)|\Big]\,dt < +\infty
$$

are fulfilled, respectively.

Theorem 2. *Let there exist a constant $\lambda \geq 1$ and a measurable function $p :]a,b[\to [0,+\infty[$ such that along with (3) the conditions*

$$
\begin{gathered}
[p_2(t)]_- = p(t)p_1^{1-\frac{1}{\lambda}}(t) \;\; \text{for} \;\; a<t<b, \\
\int\limits_a^b I_{a,b}(p_1)(t)p^\lambda(t)\,dt \leq \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta
\end{gathered} \tag{6}
$$

are satisfied. If, moreover, the functions q_1 and q_2 satisfy condition (5), then problem $(1), (2)$ has one and only one solution.

Corollary 1. *If along with* (3) *and* (5) *the condition*

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt \le \delta \tag{7}$$

holds, then problem (1), (2) *has one and only one solution.*

Corollary 2. *If along with* (3) *and* (5) *the conditions*

$$p_2(t) \ge -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t) \;\text{ for }\; a<t<b, \tag{8}$$

$$\operatorname{mes}\Big\{t \in]a,b[:\; p_2(t) > -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)\Big\} > 0 \tag{9}$$

hold, then problem (1), (2) *has one and only one solution.*

Remark 2. Inequalities (6) and (7) in Theorem 2 and Corollary 1 are unimprovable and they cannot be replaced, respectively, by the conditions

$$\int_a^b I_{a,b}(p_1)(t)p^\lambda(t)\,dt \le \Big(\frac{\pi}{\delta}\Big)^{2\lambda-2}\delta+\varepsilon$$

and

$$\int_a^b I_{a,b}(p_1)(t)[p_2(t)]_-\,dt \le \delta+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Remark 3. Inequalities (8) and (9) in Corollary 2 are unimprovable as well since if along with (3) and (5) the condition

$$p_2(t) \equiv -\Big(\frac{\pi}{\delta}\Big)^2 p_1(t)$$

holds, then problem (1), (2) either has no solution or has an infinite set of solutions.

Remark 4. Under the conditions of the above-formulated theorems and their corollaries, the function p_2 may have singularities of arbitrary order. For example, if

$$p_1(t) \equiv (t-a)^\alpha(b-t)^\beta, \quad p_2(t) \equiv \exp\Big(\frac{1}{(t-a)(b-t)}\Big),$$

$$|q_1(t)| \le (t-a)^{-2}(b-t)^{-2}\exp\Big(-\frac{1}{(t-a)(b-t)}\Big), \quad |q_2(t)| \le (t-a)^{\alpha_1}(b-t)^{\beta_1},$$

where $\alpha > -1$, $\beta > -1$, $\alpha_1 > -\alpha-2$, $\beta_1 > -\beta-2$, then the conditions of Theorems 1 and 2 as well as the conditions of Corollaries 1 and 2 are satisfied, and therefore problem (1), (2) has a unique solution.

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Existence and Stability of Uniform Attractors for N-Dimensional Impulsive-Perturbed Parabolic System

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Introduction and setting of the problem

The qualitative theory of differential equations with impulsive perturbations is outlined in [1,10,14], and for impulsive dynamical systems in [3,5,9,11,12]. In the case of an infinite dimensional phase space, the qualitative behavior of dissipative systems is studied in the framework of the theory of global attractors [15]. The generalization of the basic concepts and results of the theory of attractors to infinite-dimensional impulsive dynamical systems was carried out in [4,7,13]. The main object of research is the minimal compact uniformly attracting set – uniform attractor. The questions of existence, structure and invariance of uniform attractors for different classes of infinitely dimensional impulsive systems are dealt with in [4,6,7]. In [8], authors proposed the conditions for impulsive semiflows, which guarantee the stability of the non-impulsive part of uniform attractors. In the present paper, we refine these conditions and apply them to the study of the stability of uniform attractors of a weakly-nonlinear N-dimensional impulsive-perturbed parabolic system. More precisely, in bounded domain $\Omega\subset R^n,\ n\geq 1$ we consider the following N-dimensional weakly nonlinear parabolic system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=a_1\Delta u_1-\varepsilon f_1(u_1,\dots,u_N),\\ \dotfill\\ \dfrac{\partial u_N}{\partial t}=a_N\Delta u_N-\varepsilon f_N(u_1,\dots,u_N),\\ u_1\big|_{\partial\Omega}=\cdots=u_N\big|_{\partial\Omega}=0, \end{cases} \tag{1}$$

where $\varepsilon>0$ is a small parameter, $a_i>0$, $f=(f_1,\dots,f_N)^T$ is a nonlinear vector-function, $f\in C^1(R^2)$ satisfies

$$\exists\, C>0,\ \ \forall\, u\in R^N,\ \ \forall\, i=\overline{1,N}\quad |f_i(u)|\leq C,\ \ f'(u)\geq -C. \tag{2}$$

These assumptions guarantee global existence and uniqueness of a weak solution of the problem (1) for every initial data from the phase space $X=(L^2(\Omega))^N$ having the norm $\|u\|_X=\sqrt{\sum\limits_{i=1}^N\|u_i\|^2}$. (Here $\|\cdot\|$ and $(\cdot,\cdot)$ mean a norm and a scalar product in $L^2(\Omega)$.)

For fixed positive numbers $\alpha_1,\dots,\alpha_N$, μ and for the function $\psi\in L^2(\Omega)$ the following impulsive problem is considered: the phase point $u(t)$, when it encounters the impulse set

$$M=\Big\{u\in X \mid \sum_{i=1}^N\alpha_i(u_i,\psi)^2=1\Big\}, \tag{3}$$

is transferred to a new position $Iu \in M'$ using impulsive map $I : M \to M'$, where

$$M' = \Big\{ u \in X \mid \sum_{i=1}^{N} \alpha_i (u_i, \psi)^2 = 1 + \mu \Big\}. \tag{4}$$

It is proved in the paper that, for a sufficiently wide class of impulsive mappings $I : M \to M'$, the impulsive-perturbed problem (1)–(4) generates an impulse semiflow for sufficiently small ε generates a pulsed semiflow G_ε, which has a uniform attractor Θ_ε having an invariant and stable non-impulsive part, provided that the impulsive mapping $I : M \to M'$ is continuous.

Existence and stability of the uniform attractor of impulsive systems

Let a continuous semigroup $V : R_+ \times X \to X$, the impulsive set $M \subset X$, and the impulsive mapping $I : M \to X$ be given in the phase space $(X, \| \cdot \|_X)$. The impulsive semiflow $G : R_+ \times X \to X$ is constructed according to the following rule: [9]: if $V(t, x) \notin M$ for $x \in X$ and for all $t > 0$, then $G(t, x) = V(t, x)$; otherwise

$$G(t, x) = \begin{cases} V(t - T_n, x_n^+), & t \in [T_n, T_{n+1}), \\ x_{n+1}^+, & t = T_{n+1}, \end{cases} \tag{5}$$

where $T_0 = 0$, $T_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x_{n+1}^+ = IV(s_n, x_n^+)$, $x_0^+ = x$, s_n are the intervals between moments of impulsive perturbations characterized by the condition $V(s_n, x_n^+) \in M$.

Under conditions

$$\begin{gathered} M\text{– closed}, \ \ M \cap IM = \varnothing, \\ \forall\, x \in M, \ \ \exists\, \tau = \tau(x) > 0, \ \ \forall\, t \in (0, \tau) \quad V(t, x) \notin M, \\ \forall\, x \in X \quad t \to G(t, x) \ \text{ defined on } [0, +\infty) \end{gathered} \tag{6}$$

the formula (5) determines a semigroup $G : R_+ \times X \to X$ [3, 7], which we will call an *impulsive semiflow*.

Remark 1. It follows from conditions (6) and the continuity of the V [3, 6] that for an arbitrary $x \in X$ or there exists a moment of the time $s := s(x) > 0$ such that $\forall\, t \in (0, s) \quad V(t, x) \notin M$, $V(s, x) \in M$, or $\forall\, t > 0 \quad V(t, x) \cap M = \varnothing$ (and in this case we put $s(x) = \infty$).

Definition ([7])**.** A compact $\Theta \subset X$ will be called a uniform attractor of the impulsive semiflow G if Θ is a uniformly attracting set, i.e., for any bounded $B \subset X$

$$\operatorname{dist}(G(t, B), \Theta) \longrightarrow 0, \ \ t \to \infty,$$

and Θ is minimal among all closed uniformly attracting sets.

Remark 2. A uniform attractor may not be invariant with respect to G [7].

Lemma 1. *Suppose that a continuous semigroup $V : R_+ \times X \to X$ and a map $I : M \to X$ satisfy the following conditions: there is a compactly embedded space $Y \Subset X$ such that*

$$\begin{aligned} \exists\, C_1 > 0, \ \ \exists\, \delta > 0, \ \ \forall\, t \geq 0, \ \ \forall\, x \in X \quad & \|V(t, x)\|_X \leq \|x\|_X e^{-\delta t} + C_1, \\ \forall\, t > 0, \ \ \forall\, r > 0, \ \ \exists\, C(t, r) > 0, \ \ \forall\, x \quad & \|x\|_X \leq r, \ \ \|V(t, x)\|_Y \leq C(t, r), \\ \exists\, C_2 > 0, \ \ \forall\, x \in X \cap M \quad & \|Ix\|_X \leq \|x\|_X + C_2, \\ \forall\, r > 0, \ \ \exists\, C(r) > 0, \ \ \forall\, x \in Y \cap M \quad & \|x\|_Y \leq r, \ \ \|Ix\|_Y \leq C(r), \\ \exists\, \overline{s} > 0, \ \ \forall\, x \in IM \quad & s(x) \geq \overline{s}. \end{aligned}$$

Then the impulsive semiflow G has an uniform attractor Θ.

It is known [2,5] that one of the equivalent definitions of stability of a compact invariant set A with respect to a continuous semiflow is equality

$$A = D^{+}(A) := \bigcup_{x\in A} \big\{ y \mid \ y = \lim G(t_n, x_n), \ x_n \to x, \ t_n \geq 0 \big\}. \tag{7}$$

It was shown in [8] that a uniform attractor of an impulsive semiflow may not satisfy (7), although under additional assumptions regarding the nature of the behavior of the trajectories in the neighborhood of the impulsive set, we can obtain the following result.

Lemma 2 ([8]). *Let impulsive semiflow G has a uniform attractor Θ. Let impulsive mapping $I : M \to X$ be continuous, and the following conditions are satisfied:*

- *for any sequence $x_n \to x \in \Theta \setminus M$*

$$\begin{cases} s(x) = \infty, & \textit{if } \ s(x_n) = \infty \ \textit{ for infinitely many } n, \\ s(x_n) \to s(x), & \textit{otherwise}; \end{cases}$$

- *for any sequence $x_n \to x \in \Theta \cap M$*

$$\textit{either } \ s(x_n) = \infty \ \textit{ for infinitely many } n, \textit{ or } \ s(x_n) \to 0.$$

Then $\Theta \setminus M$ is invariant with respect to semiflow G and

$$\Theta = \overline{\Theta \setminus M}, \quad D^{+}(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{8}$$

Application to impulsive-perturbed parabolic problem

To apply Lemmas 1, 2 to impulsive problems (1)–(4), we specify the perturbation parameters. Let $\{\lambda_k\}_{k=1}^{\infty}$, $\{\psi_k\}_{k=1}^{\infty}$ be solutions to the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$. Assume that in the definition of sets M, M' we have $\psi = \psi_1$, $\lambda = \lambda_1$. Then it is natural to consider the following class of impulsive mappings $I : M \mapsto M'$:

$$\text{for } \ u = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \psi_1 + \sum_{k=2}^{\infty} \begin{pmatrix} c_1^k \\ \vdots \\ c_N^k \end{pmatrix} \psi_k \in M \quad \text{we have } \ I(u) = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \psi_1 + \sum_{k=2}^{\infty} \begin{pmatrix} c_1^k \\ \vdots \\ c_N^k \end{pmatrix} \psi_k.$$

The simplest example: $\forall\, i = \overline{1, N} \quad d_i = \sqrt{1+\mu}\, c_i$.

The main result of this paper is the following theorem.

Theorem. *Let conditions* (2) *be satisfied. Then for sufficiently small $\varepsilon > 0$, the problem* (1)*–*(4) *in the phase space $X = (L^2(\Omega))^N$ generates an impulsive semiflow having a uniform attractor Θ_ε. If, in addition, the map $I : M \mapsto M'$ is continuous, then Θ_ε has invariant non-impulsive part and satisfies the stability properties* (8)*.*

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Antiperiodic Problem with Barriers

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1 Introduction

Some real world models are described by means of impulse control of nonlinear BVPs, where time instants of impulse actions depend on intersection points of solutions with given barriers. For $i = 1, \dots, m$, and $[a, b] \subset \mathbb{R}$, continuous functions $\gamma_i : \mathbb{R} \to [a, b]$ determine barriers $\Gamma_i = \{(t, z) : t = \gamma_i(z),\ z \in \mathbb{R}\}$. A solution (x, y) of a planar BVP on $[a, b]$ is searched such that the graph of its first component $x(t)$ has exactly one intersection point with each barrier, i.e. for each $i \in \{1, \dots, m\}$ there exists a unique root $t = t_{ix} \in [a, b]$ of the equation $t = \gamma_i(x(t))$. The second component $y(t)$ of the solution has impulses (jumps) at the points $t_{1x}, \dots, t_{mx}$. Since a size of jumps and especially the points $t_{1x}, \dots, t_{mx}$ depend on x, impulses are called *state-dependent*.

More precisely, for $T > 0$ and given continuous functions $\gamma_1, \dots, \gamma_m$, we prove the existence of a T-antiperiodic solution (x, y) of the van der Pol equation with a positive parameter μ and a Lebesgue integrable T-antiperiodic function f

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \ \text{ for a.e. } \ t \in [0, T], \ \ t \notin \{t_{1x}, \dots, t_{mx}\}, \tag{1.1}$$

where y has impulses at the points $t_{1x}, \dots, t_{mx} \in (0, T)$ determined by the barriers $\Gamma_1, \dots, \Gamma_m$ through the equalities

$$t_{ix} = \gamma_i(x(t_{ix})), \ \ i = 1, \dots, m, \tag{1.2}$$

and y is continuous anywhere else in $[0, T]$. The impulse conditions have the form

$$y(t+) - y(t-) = \mathcal{J}_i(x), \ \ t = t_{ix}, \ \ i = 1, \dots, m, \tag{1.3}$$

where $\mathcal{J}_i$ are continuous bounded functionals defining a size of jumps.

Previous results in the literature for this antiperiodic problem assume that impulse points are values of given continuous functionals, see [1, 3]. Such formulation is certain handicap for applications to real world problems where impulse instants depend on barriers. We have found conditions which enable to reach such functionals from given barriers. Consequently the existence results from [2] for impulsive antiperiodic problem to the van der Pol equation formulated in terms of barriers are obtained.

Notations

- T-antiperiodic function x (satisfying (1.1), (1.2), (1.3)) will be found in the set of $2T$-periodic real-valued functions. To do it functional sets defined below are used.

- L^1 consists of $2T$-periodic Lebesgue integrable functions on $[0, 2T]$ with the norm $\|x\|_{\mathrm{L}^1} := \frac{1}{2T} \int\limits_0^{2T} |x(t)|\,\mathrm{d}t$,

- BV consists of $2T$-periodic functions of bounded variation on $[0,2T]$,
- $\operatorname{var}(x)$ for $x \in \mathrm{BV}$ is the total variation of x on $[0,2T]$,
- $\|x\|_\infty := \sup\{|x(t)| : \ t \in [0,2T]\}$ for $x \in \mathrm{BV}$,
- NBV consists of normalized functions $x \in \mathrm{BV}$ in the sense that $x(t) = \frac{1}{2}\,(x(t+) + x(t-))$,
- $\overline{x} := \frac{1}{2T} \int\limits_0^{2T} x(t)\,\mathrm{d}t = 0$ is the mean value of $x \in \mathrm{BV}$,
- $\widetilde{\mathrm{NBV}}$ consists from functions $x \in \mathrm{NBV}$ with $\overline{x} = 0$; $\widetilde{\mathrm{NBV}}$ with the norm $\operatorname{var}(x)$ is the Banach space,
- $\mathrm{AC}(J)$ consists of $2T$-periodic absolutely continuous functions on $J \subset [0,2T]$ and if $J = [0,2T]$ we write AC,
- $\widetilde{\mathrm{AC}} := \mathrm{AC} \cap \widetilde{\mathrm{NBV}}$.
- A couple $(x,y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{NBV}}$ satisfying (1.1), (1.2), (1.3) is a $2T$-periodic *solution* of problem (1.1)–(1.3). If in addition
$$x(0) = -x(T), \quad y(0) = -y(T), \tag{1.4}$$
then (x,y) is a T-*antiperiodic solution* of problem (1.1)–(1.3).

2 Main result

The main existence result from [2] is contained in the next theorem.

Theorem 2.1 (Main result)**.** *Let $T \in (0,\sqrt{3})$, $K, L \in (0,\infty)$, let $\mathcal{J}_i$, $i = 1,\dots,m$, be continuous bounded functionals on $\widetilde{\mathrm{NBV}}$, and let $f \in \mathrm{L}^1$ be T-antiperiodic, i.e. $f(t+T) = -f(t)$ for a.e. $t \in \mathbb{R}$. Assume that there exist $a, b \in (0,T)$ such that functions $\gamma_1,\dots,\gamma_m$ satisfy*
$$0 < a \le \gamma_1(z) < \gamma_2(z) < \dots < \gamma_m(z) \le b < T, \ \ z \in [-K,K]. \tag{2.1}$$
Further, assume that $L_i \in (0,1/L)$, $i = 1,\dots,m$, are such that
$$|\gamma_i(z_1) - \gamma_i(z_2)| \le L_i|z_1 - z_2|, \ \ z_1, z_2 \in [-K,K], \ \ i = 1,\dots,m. \tag{2.2}$$

Then there exists $\mu_0 > 0$ such that for each $\mu \in (0,\mu_0]$ problem (1.1)–(1.3) has a T-antiperiodic solution (x,y), where y has m jumps at the points $t_{1x},\dots,t_{mx} \in [a,b]$ and y is continuous anywhere else in $[0,T]$. Moreover, the estimate
$$|x(t)| \le \operatorname{var}(x) \le K, \ \ |y(t)| \le L, \ \ t \in [0,T], \tag{2.3}$$
is valid.

We can find the optimal (maximal) value μ_0 as follows. Since $\mathcal{J}_i$ are bounded, it holds
$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i], \ \ i = 1,\dots,m,$$
for some $a_i \in (0,\infty)$. Denote
$$c_1 := T\|f\|_{\mathrm{L}^1} + \sum_{i=1}^m a_i, \tag{2.4}$$

and define a function φ by

$$\varphi(\mu) := \frac{1-\mu T-\frac{T^2}{3}}{3}\sqrt{\frac{1-\mu T-\frac{T^2}{3}}{\mu T}}\,, \ \ \mu\in\Big(0,\frac{1}{T}-\frac{T}{3}\Big]. \tag{2.5}$$

Then, according to the proof of Theorem 2.1, $\mu_0=\varphi^{-1}(Tc_1)\in(0,\frac{1}{T}-\frac{T}{3})$.

Auxiliary results

Denote

$$(x*y)(t) := \frac{1}{2T}\int\limits_0^{2T} x(t-s)y(s)\,\mathrm{d}s, \ \ t\in[0,2T] \ \text{ for } \ x,y\in\mathrm{L}^1,$$

and remind the inequalities

$$\mathrm{var}(x*y)\le \mathrm{var}(x)\|y\|_\infty, \ \ x,y\in\mathrm{NBV}, \tag{2.6}$$

$$\mathrm{var}(x*f)\le \mathrm{var}(x)\|f\|_{\mathrm{L}^1}, \ \ x\in\mathrm{NBV}, \ f\in\mathrm{L}^1, \tag{2.7}$$

$$\|x\|_{\mathrm{L}^1}\le\|x\|_\infty\le\mathrm{var}(x), \ \ x\in\widetilde{\mathrm{NBV}}. \tag{2.8}$$

Further, using the function

$$E_1(t)=\begin{cases} T-t & \text{for } \ t\in(0,2T),\\ 0 & \text{for } \ t=0,\end{cases}$$

which fulfils

$$\mathrm{var}(E_1)=4T, \ \ \|E_1\|_\infty=T, \tag{2.9}$$

we introduce antiderivative operators I and I^2 by

$$Iu:=E_1*u\in\widetilde{\mathrm{AC}}, \ \ I^2u:=I(Iu)\in\widetilde{\mathrm{AC}}, \ \ u\in\mathrm{L}^1. \tag{2.10}$$

For $\tau\in\mathbb{R}$ we define a distribution ε_τ by the Fourirer series

$$\varepsilon_\tau := \sum_{n\in\mathbb{Z}}\big(1-(-1)^n\big)\mathrm{e}^{\frac{in\pi}{T}(t-\tau)}, \ \ t\in\mathbb{R}. \tag{2.11}$$

Then it holds

$$I\varepsilon_\tau\in\widetilde{\mathrm{NBV}}, \ \ I^2\varepsilon_\tau\in\widetilde{\mathrm{AC}}, \ \ \|I\varepsilon_\tau\|_\infty=T. \tag{2.12}$$

See [3] for more details. Using this we investigated in [3] the van del Pol equation

$$x'(t)=y(t), \quad y'(t)=\mu\Big(x(t)-\frac{x^3(t)}{3}\Big)'-x(t)+f(t) \ \text{ for a.e. } \ t\in\mathbb{R}, \tag{2.13}$$

with a positive parameter μ, a Lebesgue integrable T-antiperiodic function f, and with the state-dependent impulse conditions

$$\lim_{t\to\tau_i(x)+}y(t)-\lim_{t\to\tau_i(x)-}y(t)=\mathcal{J}_i(x), \ \ i=1,\dots,m, \tag{2.14}$$

where $\mathcal{J}_i$ and also τ_i, $i=1,\dots,m$, are given continuous and bounded real-valued functionals on $\widetilde{\mathrm{NBV}}$. For such setting we proved the existence result contained in Theorem 2.2.

Theorem 2.2 ([3, Theorem 1.1])**.** *Assume that $T\in(0,\sqrt{3})$, and the functionals $\tau_1,\dots,\tau_m$ have values in $(0,T)$. Further, let*

$$i\ne j \implies \tau_i(x)\ne\tau_j(x), \ \ x\in\widetilde{\mathrm{AC}}, \ \ i,j=1,\dots,m. \tag{2.15}$$

Then there exists $\mu_0>0$ such that for each $\mu\in(0,\mu_0]$ the problem (2.13), (2.14) *has a T-antiperiodic solution (x,y).*

3 Existence of continuous functionals

If we study problem (1.1)–(1.3) which is formulated by means of barriers, then a number of impulse points for some solution (x, y) of (1.1) is equal to a number of values of t satisfying the equations $t - \gamma_i(x(t)) = 0$, $i = 1, \dots, m$. In general, for any (x, y) satisfying (1.1), such equations need not be solvable, or they can have finite or infinite number of roots. In Theorem 2.1, we present conditions imposed on barriers which yield unique solvability of these equations provided x belongs to some suitable set Ω_{KL} (see (3.1)). This yields functionals continuous on Ω_{KL}. We prove it in the next lemmas.

In particular, for positive numbers K and L, define a set Ω_{KL}

$$\Omega_{KL} := \big\{x \in \widetilde{\mathrm{AC}} : \ \mathrm{var}(x) \le K, \ |x'(t)| \le L \text{ for a.e. } t \in [0, 2T], \ x \text{ is } T\text{-antiperiodic}\big\}. \tag{3.1}$$

Lemma 3.1. *The set Ω_{KL} is nonempty, bounded, convex and closed in $\widetilde{\mathrm{NBV}}$.*

Lemma 3.2. *Let $K, L \in (0, \infty)$. Assume that there exist $a, b \in (0, T)$ and $L_i \in (0, 1/L)$, $i = 1, \dots, m$, such that* (2.1) *and* (2.2) *are fulfilled. Then for each $x \in \Omega_{KL}$ and $i \in \{1, \dots, m\}$ the equation*

$$t = \gamma_i(x(t)) \tag{3.2}$$

has a unique solution $t_{ix} \in [a, b]$.

Lemma 3.3. *Let the assumptions of Lemma 3.2 be fulfilled. Then for $i \in \{1, \dots, m\}$, the functional*

$$\tau_i : \Omega_{KL} \to [a, b], \ \ \tau_i(x) = t_{ix}, \tag{3.3}$$

where t_{ix} is a solution of (3.2)*, is continuous.*

Having continuous functionals $\tau_1, \dots, \tau_m$ from Lemma 3.3, we can argue similarly as in [3] in the proof of Theorem 2.2 and prove Theorem 2.1.

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On Existence of Solutions with Prescribed Number of Zeros to Emden–Fowler Equations with Variable Potential

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1 Introduction

The problem of the existence of solutions to Emden–Fowler type equations with prescribed number of zeros on a given domain is studied.

Consider the equation

$$y^{(n)} + p(t, y, y', \dots, y^{n-1})|y|^k \operatorname{sgn} y = 0, \;\; k \in (0, 1) \cup (1, \infty). \tag{1.1}$$

We say that $p \in \mathfrak{P}_n$ if for some $m, M \in \mathbb{R}$ the inequalities $0 < m \le p(t, \xi_1, \xi_2, \dots, \xi_n) \le M < \infty$ hold, the function $p(t, \xi_1, \xi_2, \dots, \xi_n)$ is continuous and Lipschitz continuous in $(\xi_1, \xi_2, \dots, \xi_n)$.

We prove that this equation with $p \in \mathfrak{P}_n$ has a solution with a given finite number of zeros on a given interval. Results considering the existence of solutions with countable number of zeros are presented in [9, 10]. For the equation (1.1) with $n = 3, 4$ and constant potential $p = p_0$ the existence of solutions with a given finite number of zeros on a given interval is proved in [4], and for the case $n = 3$, $p \in \mathfrak{P}_n$ – in [5, 7]. Now we generalise this result for $n > 3$, $p \in \mathfrak{P}_n$.

2 Main result

Theorem 2.1. *For any $k \in (0, 1) \cup (1, \infty)$, $n \ge 3$, $p \in \mathfrak{P}_n$, $[a, b] \subset \mathbb{R}$, and integer $S \ge 2$, equation* (1.1) *has a solution defined on the segment $[a, b]$, vanishing on its end points a, b, and having exactly S zeros on $[a, b]$.*

3 Sketch of the proof

3.1 The case of constant potential

In the case of constant potential p proof is based on the following theorems.

Theorem 3.1 ([3], [1, Theorem 5]). *For any $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function h such that for any $p_0 \in \mathbb{R}$ with $p_0 > 0$ and any $t^* \in \mathbb{R}$ the function*

$$y(t) = |p_0|^{\frac{1}{1-k}} (t^* - t)^{-\alpha} h(\log(t^* - t)), \;\; -\infty < t < t^*, \;\; \alpha = \frac{n}{k-1}\,,$$

is a solution to equation (1.1) *with constant potential $p = p_0$.*

Theorem 3.2 ([1, Theorem 9]). *For any $n > 2$ and real $k \in (0, 1)$ there exists a non-constant oscillatory periodic function h such that for any $p_0 \in \mathbb{R}$ with $(-1)^n p_0 > 0$ and any $t^* \in \mathbb{R}$ function*

$$y(t) = |p_0|^{\frac{1}{1-k}} (t^* - t)^{-\alpha} h(\log(t^* - t)), \;\; -\infty < t < t^*, \;\; \alpha = \frac{n}{k-1},$$

is a solution to equation (1.1) *with constant potential $p = p_0$.*

Lemma 3.1 ([2, Lemma 6.1])**.** *If $y(t)$ is a solution to equation* (1.1) *with constant potential $p = p_0$, and constants A, B, C satisfy $|A| = B^{\frac{n}{k-1}}$, $B > 0$, then $z(t) = Ay(Bt + C)$ is also a solution to the same equation.*

From theorems 3.1 and 3.2 it follows that equation (1.1) with constant potential $p = p_0$ has a solution $y(t)$ with countable number of zeros. Then it is possible to choose segment $[t_1, t_2]$ where $y(t_1) = y(t_2) = 0$ and $y(t)$ has exactly S zeros on the segment. Then, due to lemma 3.1, function

$$\widetilde{y}(t) = \Big(\frac{|t_2 - t_1|}{|b - a|}\Big)^{\frac{n}{k-1}} y\Big(x_1 + \frac{|t_2 - t_1|}{|b - a|}\,(t - a)\Big), \tag{3.1}$$

is a solution to the equation, it is defined on the segment $[a, b]$, $y(a) = 0$, $y(b) = 0$, and $y(t)$ has exactly S zeros on $[a, b]$. When n is odd, we use substitution $t \mapsto -t$ to consider p_0 with opposite sign. This completes the proof in the case of constant potential.

3.2 The case of variable potential

It is impossible to use same methods to prove main theorem when $p \in \mathfrak{P}_n$. The full proof of the main theorem is given in [8] (the case $k \in (1, \infty)$) and in [6] (the case $k \in (0, 1)$). The proof is based on the following results.

Lemma 3.2 (generalisation of [2, Lemma 7.1])**.** *If $y(t)$ is a solution to* (1.1) *satisfying, at some t_0, the conditions*

$$y(t_0) \geq 0, y'(t_0) > 0, y''(t_0) \geq 0, \dots, y^{(n-1)}(t_0) \geq 0,$$

then at some $t_0' > t_0$ the solution has a local maximum and satisfies

$$\begin{aligned} t_0' - t_0 &\leq (\mu y'(t_0))^{-\frac{k-1}{k+n-1}}, \\ y(t_0') &> (\mu y'(t_0))^{\frac{n}{k+n-1}}, \end{aligned}$$

where the constant $\mu > 0$ depends only on n, k, m, M.

Lemma 3.3 (generalisation of [2, Lemma 7.2])**.** *If $y(t)$ is a solution to* (1.1) *satisfying, at some t_0', the conditions*

$$y(t_0') > 0, y'(t_0') \leq 0, \dots, y^{(n-1)}(t_0') \leq 0,$$

then at some $t_0 > t_0'$ the solution is equal to zero, and

$$\begin{aligned} t_0 - t_0' &\leq (\mu y(t_0'))^{-\frac{k-1}{n}}, \\ y'(t_0) &< -(\mu y(t_0'))^{\frac{k+n-1}{n}}, \end{aligned}$$

where the constant $\mu > 0$ depends only on n, k, m, M.

Lemma 3.4 (generalisation of [2, Lemma 7.3])**.** *Under the assumptions of Lemmas 3.2 and 3.3, for any $t_1 > t_0$ with $y(t_0) = 0$, $y(t_1) = 0$ the inequality*

$$|y'(t_1)| > Q|y'(t_0)|$$

holds true, where the constant $Q > 1$ depends only on k, m, M.

Lemma 3.5 ([5, 8])**.** *Suppose $D \subset \mathbb{R}^n$ and $\widetilde{D} \subset \mathbb{R}^{n+1}$ are open connected sets such that for every $c \in D$ there exists a segment $[0, x_c]$ with $[0, x_c] \times \{c\} \subset \widetilde{D}$. Suppose that $f(x, c)$ is a continuous function $\widetilde{D} \to \mathbb{R}$ as well as its derivative in x. Suppose that for every $c \in D$ the following conditions are fulfilled.*

- *$f(0,c)=0$.*
- *There exists $x_1(c)\in(0,x_c)$ such that $f(x_1(c),c)=0$ and $f(x,c)\neq 0$ for all $x\in(0,x_1(c))$.*
- $$f'_x(x,c)\big|_{x=0}\neq 0,\quad f'_x(x,c)\big|_{x=x_1(c)}\neq 0.$$

Then $x_1(c)$ is a continuous function $D\to\mathbb{R}$.

In the case $k>1$ the main result is proved as follows. We consider a solution $y(t)$ with initial values

$$y(a)=0, y'(a)=y_1, y''(a)=y_2,\dots,y^{(n-1)}(a)=y_{n-1},$$

where $y_i>0$, $i=1,\dots,n-1$. Due to Lemmas 3.2–3.4, the solution $y(t)$ oscillates; so, $y(t)$ has a sequence of zeros t_j, $j\in\mathbb{N}$. We consider the position of a particular zero t_{S-1} as a function of initial values $y_1,\dots,y_{n-1}$, and with the help of Lemma 3.5 we find out that this function is continuous. Then, obtaining some estimates, we prove that the range of values of $t_{S-1}(y_1,\dots,y_{n-1})$ is $(a,+\infty)$, and that means that for some initial values we have $t_{S-1}=b$, whence the corresponding solution $y(t)$ has exactly S zeros on $[a,b]$.

In the case $k\in(0,1)$ the same methods apply, but equation (1.1) with $k\in(0,1)$ does not satisfy the conditions of the theorem of continuous dependence of solutions to ODE, which was used in the proof. We have to find a workaround here, and it is provided by the following lemmas (see [6]), which act as replacements for the mentioned continuous dependence theorem.

Lemma 3.6 ([6])**.** *Suppose that $n\geq 3$, $k\in(0,1)$, $p\in\mathfrak{P}_n$, and y is a solution to*

$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ y^{(i)}(t_0)=y_i,\ \ i=\overline{0,n-1},$$

defined on $[a,b]$. In addition, suppose that for some $w\in\mathbb{R}$ the inequality $|y'|\geq w>0$ holds true on $[a,b]$. Then there exists $v\in\mathbb{R}^+$ such that for every $I=[t_0,t^*]\subset[a,b]$ with $|I|<v$, for every $\varepsilon>0$ there exists $\delta>0$ such that if some $q\in\mathfrak{P}_n$, $z_i\in\mathbb{R}$, $i=\overline{0,n-1}$, satisfy the inequalities

$$|p-q|<\delta,\ \ |z_i-y_i|<\delta,\ \ i=\overline{0,n-1},$$

and z is a solution to

$$z^{(n)}+q(t,z,z',\dots,z^{(n-1)})|z|^k\operatorname{sgn}z=0,\ \ z^{(i)}(t_0)=z_i,\ \ i=\overline{0,n-1},$$

then z is defined on or can be extended onto I with the inequalities

$$\big|z^{(i)}(t)-y^{(i)}(t)\big|<\varepsilon,\ \ i=\overline{0,n-1},$$

satisfied on it.

Lemma 3.7 ([6])**.** *Suppose that $n\geq 3$, $k\in(0,1)$, $p\in\mathfrak{P}_n$, and y is a solution to*

$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ y^{(i)}(t_0)=y_i,\ \ i=\overline{0,n-1},$$

defined on $[a,c]$, and y has a finite number of zeros, all of them being of first order. Then for every $\varepsilon>0$ there exists $\delta>0$ such that if some $q\in\mathfrak{P}_n$, $z_i\in\mathbb{R}$, $i=\overline{0,n-1}$, satisfy the inequalities

$$|p-q|<\delta,\ \ |z_i-y_i|<\delta,\ \ i=\overline{0,n-1},$$

and z is a solution to

$$z^{(n)}+q(t,z,z',\dots,z^{(n-1)})|z|^k\operatorname{sgn}z=0,\ \ z^{(i)}(t_0)=z_i,\ \ i=\overline{0,n-1},$$

then z is defined on or can be extended onto $[a,c]$ with the inequalities

$$\big|z^{(i)}(t)-y^{(i)}(t)\big|<\varepsilon,\ \ i=\overline{0,n-1},$$

satisfied on it.

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Investigation of Periodic Solutions of Autonomous System by Halving the Interval

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We study the T-periodic boundary value problem for the autonomous system of differential equations

$$u'(t)=f(u(t)),\ \ t\in[0,T];\quad u(0)=u(T), \tag{1}$$

where T is the unknown period, $f:\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function defined on a closed bounded set (see (9)).

In [3] , we have suggested an approach for the investigation of general type of non-linear boundary value problem with the functional boundary conditions

$$u'(t)=f(t,u(t)),\ \ t\in[a,b],\ \ \Phi(u)=d, \tag{2}$$

where $\phi:C([a,b],\mathbb{R}^n)$ is a vector functional (possibly non-linear), which involves a kind of reduction to a parametrized family of problems with separated conditions

$$u'(t)=f(t,u(t)),\ \ t\in[a,b], \tag{3}$$

$$u(a)=z,\quad u(b)=\eta, \tag{4}$$

where $z:=\operatorname{col}(z_1,\dots,z_n)$, $\eta:=\operatorname{col}(\eta_1,\dots,\eta_n)$ are unknown parameters. The techniques of [3] are based on properties of the iteration sequence

$$\begin{aligned}
u_m(t,z,\eta)&:=z+\int_a^t f(s,u_{m-1}(s,z,\eta))\,ds\\
&\qquad-\frac{t-a}{b-a}\int_a^b f(s,u_{m-1}(s,z,\eta))\,ds+\frac{t-a}{b-a}\,[\eta-z],\ \ t\in[a,b],\ \ m=1,2,\dots, \qquad (5)\\
u_0(t,z,\eta)&:=z+\frac{t-a}{b-a}\,[\eta-z]
\end{aligned}$$

and on the solution of the algebraic system

$$\Delta(z,\eta) := \eta - z - \int\limits_a^b f(s, u_m(s,z,\eta))\,ds. \tag{6}$$

Formulas (5), (6) are used to compute the corresponding functions explicitly for certain values of m, which, under additional conditions, allows one to prove the solvability of the problem and construct approximate solutions.

It is known, that the T-periodic solution $u^*(t)$ of autonomous system is not isolated in the extended phase space which means that every member of the one-parameter family of functions $t \to u^*(t+\varphi)$, $\varphi \in [0,T]$ is also a T-periodic solution. But, all these periodic solutions represent one and the same trajectory. In the autonomous T-periodic case (1) $z = \eta$ and the direct application of the successive approximation technique (5), (6) implies that

$$u_m(t,z,\eta) = z, \quad \Delta(z,\eta) = f(z) = 0.$$

Therefore, the successive approximations scheme determined by (5), (6) "detects" only constant stationary periodic solutions. In [1], it was considered the investigation of periodic solutions of autonomous systems by transforming them with special replacements into non-autonomous systems. Here we show that for the study of periodic solutions of autonomous systems it is advisable to use the technique of dividing a segment in half [2,4].

In view of the foregoing, without loss of generality, having replaced u^* by $u^*(\,\cdot\,+\varphi)$ with a suitable φ, we can assume in the subsequent consideration that a certain fixed, say jth component of the periodic function $u^*(\,\cdot\,+\varphi)$ takes extremal value over $[0,T]$ at the point $t=0$. So, we study the periodic solution of (1) for which

$$f_j\big(u_1(0), u_{21}(0), \dots, u_{n1}(0)\big) = 0. \tag{7}$$

Let us fix certain closed bounded sets $D_0, D_1 \subset \mathbb{R}^n$ and focus on the continuously differentiable T-periodic solutions u (T is unknown) of problem (1), (7) with values

$$u(0) \in D_0, \;\; u(T/2) \in D_1, \;\; u(T) \in D_0. \tag{8}$$

Based on the sets D_0 and D_1, we introduce the sets

$$D_{0,1} = (1-\theta)z + \theta\eta, \;\; z \in D_0, \;\; \eta \in D_1, \;\; \theta \in [0,1],$$

and its component-wise vector ρ neighborhood

$$D_\rho = O(D_{0,1}, \rho). \tag{9}$$

The problem is to find a continuously differentiable solution $u : [0,T] \to D_\rho$ to problem (1) for which inclusions (8) hold. We introduce the vectors of parameters

$$z = \operatorname{col}(z_1, z_2, \dots, z_n), \quad \eta = \operatorname{col}(\eta_1, \eta_2, \dots, \eta_n)$$

by formally putting

$$z := u(0), \quad \eta := u(T/2), \quad z := u(T). \tag{10}$$

Instead of (1) using a natural interval halving technique, we will consider on the intervals $t \in [0,T/2]$ and $[T/2,T]$, respectively, the following two "model-type" two-point problems with separated parametrised conditions

$$x'(t) = f(x(t)), \;\; t \in [0,T/2], \;\; x(0) = z, \;\; x(T/2) = \eta, \tag{11}$$

$$y'(t) = f(y(t)), \;\; t \in [T/2,T], \;\; y(T/2) = \eta, \;\; y(T) = z. \tag{12}$$

We suppose that

$$f \in \operatorname{Lip}(K, D_\rho) \ \text{ with the vector } \rho \text{ satisfying the inequality } \ \rho \geq \frac{T}{2}\,\delta_{D_\rho}(f), \tag{13}$$
$$r(Q) < 1, \ \text{ where } \ Q = \frac{3T}{20}\,K, \ \ \delta_{D_\rho}(f) = \frac{1}{2}\Big(\max_{x\in D_\rho} f(x) - \min_{x\in D_\rho} f(x)\Big).$$

To study the solutions of problems (11) and (12) let us introduce the following parametrised sequence of functions

$$\begin{aligned} x_m(t,z,\eta,T) &:= z + \int\limits_0^t f(x_{m-1}(s,z,\eta,T))\,ds \\ &\quad - \frac{2t-}{T}\int\limits_0^{T/2} f(x_{m-1}(s,z,\eta,T))\,ds + \frac{2t}{T}\,[\eta - z], \ \ t \in [0,T/2], \ \ m = 1,2,\dots, \\ x_0(t,z,\eta,T) &:= z + \frac{2t}{T}\,[\eta - z], \end{aligned} \tag{14}$$

and

$$\begin{aligned} y_m(t,z,\eta,T) &:= \eta + \int\limits_{T/2}^t f(y_{m-1}(s,z,\eta,T))\,ds - \frac{2(t-T/2)}{T}\int\limits_{T/2}^{T} f(y_{m-1}(s,z,\eta,T))\,ds \\ &\quad + \frac{2(t-T/2)}{T}\,[z-\eta], \ \ t \in [T/2,T], \ \ m = 1,2,\dots, \\ y_0(t,z,\eta,T) &:= \eta + \frac{2(t-T/2)}{T}\,[\eta - z]. \end{aligned} \tag{15}$$

Theorem 1. *Assume that for problem* (1) *conditions* (13) *are satisfied. Then for arbitrary* $(z,\eta) \in D_0 \times D_1$*:*

1. *All members of sequences* (14), (15) *are continuously differentiable functions on the intervals* $t \in [0,T/2]$ *and* $t \in [T/2,T]$ *satisfying conditions*
$$x_m(0,z,\eta) = z, \quad x_m(T/2,z,\eta) = \eta, \quad y_m(T/2,z,\eta) = \eta, \quad y_m(T,z,\eta) = z.$$
2. *Sequences* (14), (15) *in* $t \in [0,T/2]$ *and* $t \in [T/2,T]$*, respectively, converge uniformly as* $m \to \infty$ *to the limit functions*
$$x_\infty(t,z,\eta,T) = \lim_{m\to\infty} x_m(t,z,\eta,T), \quad y_\infty(t,z,\eta,T) = \lim_{m\to\infty} y_m(t,z,\eta,T).$$
3. *The limit functions are the unique continuously differentiable solution of the following additively perturbed equations for all* $(z,\eta) \in D_0 \times D_1$
$$x(t) := z + \int\limits_0^t f(x(s))\,ds - \frac{2t-}{T}\int\limits_0^{T/2} f(x(s))\,ds + \frac{2t}{T}\,[\eta - z], \ \ t \in [0,T/2],$$
$$y(t) := \eta + \int\limits_{T/2}^t f(y(s))\,ds - \frac{2(t-T/2)}{T}\int\limits_{T/2}^{T} f(y(s))\,ds + \frac{2(t-T/2)}{T}\,[z-\eta], \ \ t \in [T/2,T].$$

Theorem 2. *Let the conditions of Theorem 1 hold. Then the function*

$$u_\infty(t)=\begin{cases} x_\infty(t,z,\eta), & t\in[0,T/2]\\ x_\infty(t,z,\eta), & t\in[T/2,T]\end{cases}$$

is a continuously differentiable T-periodic solution of (1) *if and only if the triplet (z,η,T) satisfies the system of $2n+1$ algebraic or transcendental determining equations*

$$\Delta(z,\eta,T)=\eta-z-\int\limits_0^{T/2} f(x_\infty(s,z,\eta,T))\,ds=0, \tag{16}$$

$$H(z,\eta,T)=\eta-z-\int\limits_{T/2}^{T} f(y_\infty(s,z,\eta,T))\,ds=0,$$

$$f_j(z_1,z_2,\dots,z_n)=0.$$

Note that the solvability of (1) *can be established by studying the approximate determining system, when in* (16) *instead of ∞ stands m.*

Let us apply the approach described above to the system

$$\begin{aligned}\frac{du_1}{dt}&=u_2,\\ \frac{du_2}{dt}&=-4u_1+u_1^2+\frac{u_2^2}{16}-\frac{1}{64}\,.\end{aligned}\tag{17}$$

The domains D_0, D_1, vector ρ can be choosen to satisfy the conditions of Theorem 1. Applying Maple (14), we carried out the calculations. Note that as a zeroth approximation in formulas (14), (15), one can choose any function with values in domain D_ρ.

Introduce the following parameters $z=\operatorname{col}(z_1,z_2),\eta=\operatorname{col}(\eta_1,\eta_2)$. If in (7) $j=1$, then from (17) it follows that $z_2=0$. The system (17) has two stationary constant solutions

$$z_1=-0.9765029026\cdot10^{-3},\ \ z_2=0\ \text{ and }\ z_1=16.00097650,\ \ z_2=0.$$

The exact $\frac{\pi}{2}$-periodic solution of system (17) is $u_1(t)=\frac18\,\cos(4t)$, $u_2(t)=-\frac12\,\sin(4t)$. For a different number of approximations m, we obtain from (14), (15) and from the approximate determining system (16) the following numerical values for the introduced parameters which are presented in Table 1.

Table 1.

m	z_1	η_1	η_2	T
0	0.09964844522	-0.1003515548	$1.079348881\cdot10^{-12}$	1.570796327
1	0.09965288938	-0.1003558995	$3.849442526\cdot10^{-12}$	1.570796327
3	0.0996603478	-0.1003631726	$-1.637826662\cdot10^{-12}$	1.570796327
Exact	0.125	-0.1250000000	0	1.570796327

Note that the second equilibrium point and the $\frac{\pi}{2}$-periodic solution are unstable.

On Figure 1, we have the graphs of the exact solution (solid line) and its third approximation ($\times$) for the first and second components on the intervals $t\in[0,T]$.

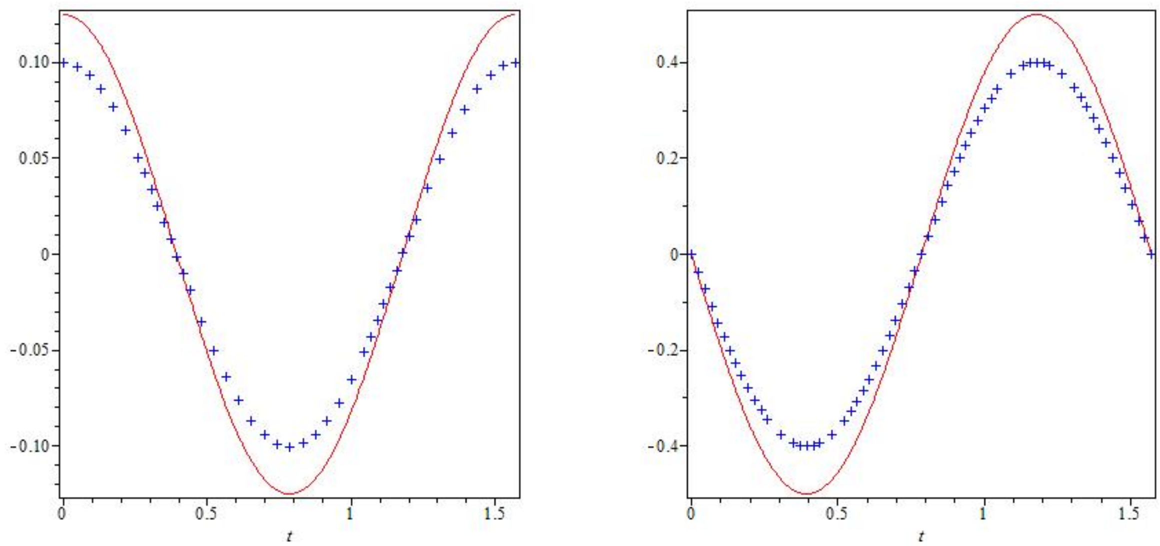

Figure 1.

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Definition and Properties of Perron Stability of Differential Systems

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1 The Perron stability definition

For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider the system

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \;\; t \in \mathbb{R}^+ \equiv [0,\infty), \;\; x \in G, \tag{1.1}$$

with the right-hand side $f \in C^1(\mathbb{R}^+ \times G)$ admitting a zero solution. Let $\mathcal{S}_*(f)$ denote the set of all non-continuable nonzero solutions x of the system (1.1), then let $\mathcal{S}_\delta(f)$ and $\mathcal{S}^\delta(f)$ denote its subsets given by the initial conditions $|x(0)| < \delta$ and $|x(0)| = \delta$, respectively.

Definition 1.1. We say that a system (1.1) (more precisely, its zero solution, implied implicitly everywhere below) has the following *Perron features*:

(1) *Perron stability* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement
$$\varliminf_{t\to\infty} |x(t)| < \varepsilon; \tag{1.2}$$

(2) *asymptotic Perron stability* if there is a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement
$$\varliminf_{t\to\infty} |x(t)| = 0; \tag{1.3}$$

(3) *Perron instability* if there is no Perron stability, i.e. there is an $\varepsilon > 0$ such that for any $\delta > 0$ there is a solution $x \in \mathcal{S}_\delta(f)$ not satisfying the requirement (1.2) (in particular, not defined on the whole semi-axis $\mathbb{R}^+$);

(4) *complete Perron instability* if there are $\varepsilon, \delta > 0$ such that no solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement (1.2).

Remark 1.1. In Definition 1.1, each of the four Perron features:

(a) in a standard way (namely, with a simple shift of coordinates) extends from the *zero* solution to any other one, and not only to the points of rest of the system under study;

(b) is of a *local* character, i.e. it depends on the behavior of only those solutions that start near zero;

(c) characterizes the behavior of solutions starting near zero from the point of view of the possibility for them to *approach* the origin arbitrarily *late* or, conversely, *ultimately move away* from it.

The next two theorems describe some seemingly paradoxical situations.

Theorem 1.1. *There is a complete Perron unstable two-dimensional system* (1.1) *which has at least one solution $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.3) *and even the condition*

$$\lim_{t\to\infty} |x(t)| = 0. \tag{1.4}$$

Theorem 1.2. *There exists a Perron unstable two-dimensional autonomous system* (1.1) *such that for some $\delta > 0$ all solutions $x \in \mathcal{S}^\delta(f)$ satisfy the requirement* (1.4).

2 Perron and Lyapunov stability joint properties

Definition 2.1 ([1, Ch. II, § 1])**.** Let us assign the *Lyapunov analogue* to each of the four Perron features above:

(a) *Lyapunov stability*, *instability* and *complete instability* are obtained by replacing the requirement (1.2) in the first, third and fourth paragraphs of the Definition 1.1 respectively by the following requirement

$$\sup_{t\in\mathbb{R}^+} |x(t)| < \varepsilon;$$

(b) *asymptotic Lyapunov stability* is obtained by replacing the requirement (1.3) in the second paragraph of the Definition 1.1 by the requirement (1.4), but with the Lyapunov stability.

Remark 2.1. For any system (1.1) the following *logical* statements are true:

(1) it is either Perron (Lyapunov) stable, or Perron (respectively, Lyapunov) unstable;

(2) if it is asymptotically Perron (Lyapunov) stable, then it is Perron (respectively, Lyapunov) stable;

(3) if it is completely Perron (Lyapunov) unstable, then it is Perron (respectively, Lyapunov) unstable;

(4) if it is Lyapunov stable (asymptotically), then it is Perron stable (respectively, asymptotically);

(5) if it is Perron unstable (completely), then it is Lyapunov unstable (respectively, completely).

Definition 2.2. We will call *strict* the following varieties of Perron (Lyapunov) features:

(a) asymptotic Perron (Lyapunov) stability;

(b) *non-asymptotic* Perron (Lyapunov) stability;

(c) complete Perron (Lyapunov) instability;

(d) *incomplete* Perron (Lyapunov) instability.

Consider a *linear* system of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \tag{2.1}$$

defined by its continuous operator function $A : \mathbb{R}^+ \to \operatorname{End}\mathbb{R}^n$ (if it is bounded, we call the system *bounded* too). Denote by $\mathcal{S}_A^\delta$ the set of solutions x of the system (2.1) satisfying the initial condition $|x(0)| = \delta$.

All combinations of varieties of stability features from the Definition 2.2 which are logically admissible by the formulation of the previous remark turn out to be possible.

Theorem 2.1. *Any pair formed by any strict Perron and Lyapunov features and not conflicting with the statements of the Remark 2.1 is implemented in some at least two-dimensional bounded linear system* (2.1)*.*

A special role in the study on the stability of a linear (and not only) system is played by *characteristic exponents* of its solutions $x \in \mathcal{S}_*(f)$ – the *Lyapunov* ones [2, Ch. I] and, respectively, the *Perron* ones [3, § 2]

$$\lambda(x) \equiv \varlimsup_{t\to\infty} \frac{1}{t}\,\ln|x(t)|, \quad \pi(x) \equiv \varliminf_{t\to\infty} \frac{1}{t}\,\ln|x(t)|.$$

Theorem 2.2. *For each $n \in \mathbb{N}$ there is a complete Lyapunov unstable, but asymptotically (non-asymptotically) Perron stable n-dimensional bounded linear system* (2.1) *for which all Lyapunov exponents are positive and all Perron exponents are negative (respectively, equal to zero).*

From a practical point of view, the following two most *natural* situations seem to be particularly important:

(1) asymptotic Perron stability combined with Lyapunov stability;

(2) complete Perron (and, therefore, Lyapunov) instability.

3 The important special cases

If the system (1.1) is *one-dimensional*, then the verification of Perron features is somewhat simplified because of the possibility to order the solutions by increasing their initial values in the *numerical* phase straight line.

Theorem 3.1. *For a one-dimensional system* (1.1)*:*

(1) *Perron stability is equivalent to the fact that for any $\varepsilon > 0$ there exist two opposite-sign solutions $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.2)*;*

(2) *asymptotic Perron stability is equivalent to the existence of two opposite-sign solutions $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.3)*;*

(3) *complete Perron instability is equivalent to the existence of an $\varepsilon > 0$ such that for any $\delta > 0$ there are two opposite-sign solutions $x \in \mathcal{S}_\delta(f)$ that do not satisfy the requirement* (1.2)*.*

Remark 3.1. In the case of complete Perron instability, it is fundamentally excluded (due to the continuous dependence of the solutions on the initial values) the opportunity to find $\varepsilon, \delta > 0$, and $T \in \mathbb{R}$ such that *all at once* solutions $x \in \mathcal{S}_\delta(f)$ satisfy the requirement

$$\inf_{t\,T} |x(t)|\,\varepsilon. \tag{3.1}$$

Despite the Remark 3.1, in both one-dimensional and *autonomous* cases, the situation described in Theorem 1.1 is *impossible*, and the complete Perron instability still has a certain *uniformity.*

Theorem 3.2. *If a one-dimensional or autonomous system* (1.1) *is completely Perron unstable, then:*

(1) *for some $\varepsilon > 0$ no solution $x \in \mathcal{S}_*(f)$ satisfies the requirement* (1.2)*;*

(2) *for any $\delta > 0$ there exists an $\varepsilon > 0$ such that all solutions $x \in \mathcal{S}_*(f) \setminus \mathcal{S}_\delta(f)$ satisfy the requirement* (3.1) *already at $T = 0$.*

Each of the Perron features in the case of a *linear* system is completely determined by the properties of its solutions starting on some *sphere.*

Theorem 3.3. *The Perron stability of the linear system* (2.1) *is equivalent to fulfilling the requirement*

$$\sup_{x\in\mathcal{S}_A^1}\varliminf_{t\to\infty}|x(t)|<\infty,$$

and its asymptotic Perron stability or complete Perron instability is equivalent to the fact that any solution $x\in\mathcal{S}_A^1$ satisfies the requirement (1.3) *or, respectively, the requirement*

$$\lim_{t\to\infty}|x(t)|=\infty. \tag{3.2}$$

In the simplest case of a *linear autonomous* system the Perron and Lyapunov stability analysis lead to the *identical* result (unambiguously recognized by the real parts of the eigenvalues of the operator that defines the system and the orders of its Jordan cells corresponding to the purely imaginary ones [1, Ch. II, § 8]).

Theorem 3.4. *The linear autonomous system* (2.1) *is Perron stable (asymptotically stable, unstable, completely unstable) if and only if it is Lyapunov stable (respectively, asymptotically stable, unstable, completely unstable).*

The statement of Theorem 3.4 does *not extend* from autonomous linear systems to a slightly wider class of *regular* linear systems [1, Ch. III, § 11].

Theorem 3.5. *For each $n\in\mathbb{N}$ there exists a regular bounded linear system* (2.1) *that is asymptotically Perron stable, but completely Lyapunov unstable.*

In the case of a *linear* system, the fulfillment of the requirements (1.3) or (3.2) not for all its non-zero solutions, but only for those that constitute a *fundamental* solution system is not sufficient for Perron stability or, respectively, complete Perron instability.

Theorem 3.6. *For each $n>1$, there is an n-dimensional bounded linear system* (2.1) *with Perron instability (with incomplete instability) for which the Perron exponents of all solutions from some of its fundamental systems are negative (respectively, positive).*

However, in some (even non-linear) cases, the knowledge of the *set of exponents of all* solutions of the system starting close to zero gives full information about the Perron and Lyapunov features.

Theorem 3.7. *If for some $\delta>0$ the Perron (Lyapunov) exponents of all solutions $x\in\mathcal{S}_\delta(f)$ of the system* (1.1) *are negative, then the system is asymptotically Perron (respectively, Lyapunov) stable, and if they are positive, then it is completely unstable.*

4 The first-order stability

Let the *linear part* be distinguished in the right-hand side of the system (1.1), i.e. let it be represented as

$$\dot{x}=A(t)x+h(t,x)\equiv f(t,x),\ \ (t,x)\in\mathbb{R}^+\times G,\quad \sup_{t\in\mathbb{R}^+}|h(t,x)|=o(x),\ \ x\to 0, \tag{4.1}$$

where $A(t)\equiv f'_x(t,0)$, $t\in\mathbb{R}^+$. Then for it the corresponding system (2.1) will be considered as the *first approximation* system.

Definition 4.1. We say that the first approximation system (2.1) *provides* a given Perron or Lyapunov feature if any system (4.1) with this first approximation has the given one.

The study of asymptotic stability by the first approximation, which is the essence of the first Lyapunov method, has been the subject of a huge number of works (see [3, § 11]). However, the study by the first approximation of stability or asymptotic stability, according to Perron or Lyapunov – all of them are possible only for *the same* systems.

Theorem 4.1. *If a linear approximation* (2.1) *provides at least one of the four features: Perron stability, Lyapunov stability, asymptotic Perron stability, or asymptotic Lyapunov stability – then it provides the other three of them.*

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Asymptotic Behavior of Solutions of Third Order Ordinary Differential Equations

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We consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^{\sigma}, \tag{1}$$

where $\alpha_0 \in \{-1;1\}$, $p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\sigma \in \mathbb{R}$, $\infty < a < \omega \le +\infty$. It belongs to the equations class of the form

$$y''' = \alpha_0 p(t) L(y), \tag{2}$$

where $\alpha_0 \in \{-1;1\}$,$p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\infty < a < \omega \le +\infty$, the function L is continuous and positive in a one-sided neighborhood of Δ_{Y_0} at points Y_0 (Y_0 equals $\pm\infty$).

For equations of the form (2) in the work of N. Sharay and V. Evtukhov [4] for the function $L(y)$ with rapidly varying nonlinearity it was investigated the question of the existence and asymptotic behavior as $t \to \omega$ of the so-called $P_\omega(Y_0,\lambda_0)$-solution.

In [5, 10] A. Stekhun and V. Evtukhov obtained the results on the existence and asymptotic behavior as $t \to \omega$ of the endangered and unlimited solutions of the differential equation (2), where $L(y) = yL_1(y)$, $L_1(y)$ is a regularly varying function.

For second order equations of the form (1) in the works of V. Evtukhov and M. Jaber [1,2] it was investigated the question on the existence and asymptotic behavior as $t \uparrow \omega$ of all $P_\omega(\lambda_0)$-solutions. It seems natural to try to extend these results to the third-order differential equations.

A solution y of equation (1), specified on the interval $[t_y, w) \subset [a,\omega)$, is said to be a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty \end{cases} (k=0,1,2), \quad \lim_{t\uparrow\omega} \frac{[y''(t)]^2}{y'''(t)y'(t)} = \lambda_0.$$

In the work [3] it is shown that a set of $P_\omega(\lambda_0)$-solutions with regards to their asymptotic properties in the five class solutions, corresponding values $\lambda_0 \in \mathbb{R}\backslash\{0;1;\frac{1}{2}\}$, $\lambda_0 = \pm\infty$, $\lambda_0 = 0$, $\lambda_0 = \frac{1}{2}$ and $\lambda_0 = 1$.

Earlier in [7–9] the results were obtained in the case, when $\lambda_0 \in R\backslash\{0,\pm1,\frac{1}{2}\}$ and $\lambda_0 = \pm\infty$. The goal of the work is the establishment existence conditions for equation (1) of $P_\omega(1)$-solutions and also asymptotic representations as $t \uparrow \omega$ of such solutions and their derivatives of second order.

We introduce the necessary notation.

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}\quad I_B(t)=\int\limits_B^t p^{\frac13}(\tau)\,d\tau,\quad B=\begin{cases} a & \text{if } \int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega p^{\frac13}(\tau)\,d\tau<+\infty.\end{cases}$$

Theorem 1. *Let $\sigma\neq 3$, the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{t\uparrow\omega}\frac{(p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{2-\sigma}})'}{p^{\frac23}(t)|I_B(t)|^{\frac{2\sigma}{3-\sigma}}}. \tag{3}$$

For the existence of $P_\omega(1)$-solutions of equation (1) *it is necessary and sufficient the conditions*

$$\alpha_0>0 \ \textit{ and } \ \lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{3-\sigma}}=\infty \tag{4}$$

to hold. Moreover, for each such solution there take place the following asymptotic representations as $t\uparrow\omega$

$$\ln|y(t)|=\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{3}{3-\sigma}}[1+o(1)],\quad \frac{y'(t)}{y(t)}=p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}[1+o(1)],$$
$$\frac{y''(t)}{y(t)}=p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}[1+o(1)],$$

where $\mu=\operatorname{sign}(\frac{3-\sigma}{3}I_B(t))$.

Theorem 2. *Let $\sigma\neq 3$, the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and along with* (3), (4) *the following condition*

$$\lim_{t\uparrow\omega}\frac{(p^{\frac23}(t)|I_B(t)|^{\frac{2\sigma}{3-\sigma}})'}{p(t)|I_B(t)|^{\frac{3(\sigma-1)}{3-\sigma}}}=0$$

hold. Then for any $C=\pm1$ equation (1) *has a $P_\omega(1)$-solution. Furthermore, for every such solution the following asymptotic representations as $t\to\omega$*

$$y(t)=C\exp\Big[\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{3}{3-\sigma}}\Big][1+o(1)],\quad y'(t)=\mu p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}y(t)[1+o(1)],$$
$$y''(t)=\mu p^{\frac23}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{2\sigma}{3-\sigma}}y(t)[1+o(1)]$$

take place.

We give a corollary of these theorems, when $\sigma=0$, i.e. for the following linear differential equation

$$y'''=\alpha_0 p(t)y, \tag{5}$$

where $\alpha_0\in\{-1;1\}$, $\sigma\in\mathbb{R}$, $p:[a,w)\to(0,+\infty)$ is a continuous function, $a<w\leq+\infty$.

Corollary 1. *Let the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and there exist a finite or equal to $\pm\infty$ limit $\lim\limits_{t\uparrow\omega} p'(t)p^{-\frac{5}{3}}(t)$. For the existence of $P_\omega(1)$-solutions of equation* (5) *it is necessary and sufficient the conditions*

$$\alpha_0>0 \ \ and \ \ \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)=+\infty \tag{6}$$

to hold. Moreover, for each such solution the following asymptotic representations as $t\uparrow\omega$

$$\ln|y(t)|=\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|[1+o(1)],\quad \frac{y'(t)}{y(t)}=p^{\frac{1}{3}}(t)[1+o(1)],\quad \frac{y''(t)}{y(t)}=p^{\frac{1}{3}}(t)[1+o(1)],$$

where $\mu=\operatorname{sign}(I_B(t))$, take place.

Corollary 2. *Let the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and along with conditions* (6) *the following condition is satisfied*

$$\int_a^\omega\Big|\frac{p'(t)}{p(t)}\Big|\,dt<+\infty.$$

Then equation (5) *has a $P_\omega(1)$-solution. Furthermore, for any such solution the following asymptotic representations as $t\to\omega$*

$$y_i(t)=\exp\big[(-1)^{i-1}I_B(t)\big][1+o(1)],\quad y'(t)=(-1)^{i-1}p^{\frac{1}{3}}(t)y(t)[1+o(1)],$$
$$y''(t)=(-1)^{i-1}p^{\frac{2}{3}}(t)y(t)[1+o(1)]\ \ (i=1,2,3)$$

take place.

The obtained results are consistent with the already known results for linear differential equations (see [6, Chapter 1]).

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On Increasing the Order of Smallness of Fast Variables in Linear Differential Systems

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Let

$$G=\big\{t,\varepsilon:\ t\in[t_0,+\infty),\varepsilon\in(0,\varepsilon_0),\ \varepsilon_0\in\mathbf{R}^+\big\}.$$

Definition. We say that the function $f(t,\varepsilon)$ belongs to the class $S(m)$, $m\in\mathbf{N}\cup\{\mathbf{0}\}$, if:

1) $f:G\to\mathbf{C}$,
2) $f(t,\varepsilon)\in C^m(G)$ at t,
3) $d^kf(t,\varepsilon)/dt^k=\varepsilon^kf_k(t,\varepsilon)$ $(0\le k\le m)$,

$$\|f\|_{S(m)}\overset{def}{=}\sum_{k=0}^{m}\sup_G|f_k(t,\varepsilon)|<+\infty.$$

By slowly varying function we mean a function from the class $S(m)$.

Consider the system of differential equations

$$\frac{dx}{dt}=\Big(A_0(t,\varepsilon)+\sum_{s=1}^{r}A_s\big(t,\varepsilon,\theta(t,\varepsilon)\big)\big(\mu(\theta(t,\varepsilon))\big)^s\Big)x, \tag{1}$$

$x=\operatorname{colon}(x_1,\dots,x_n)$, $A_0(t,\varepsilon)$ – $(N\times N)$-matrix, whose elements belong to the class $S(m)$. The function $\theta(t,\varepsilon)$ has the form

$$\theta(t,\varepsilon)=\int_{t_0}^{t}\varphi(\tau,\varepsilon)\,d\tau, \tag{2}$$

$\varphi\in\mathbf{R}^+$, $\varphi(t,\varepsilon)\in S(m)$, $\inf\limits_G\varphi(t,\varepsilon)=\varphi_0>0$. The elements of matrices $A_s(t,\varepsilon,\theta)$ belong to the class $S(m)$ with respect to t, ε, are continuous and 2π-periodic with respect to $\theta\in[0,+\infty)$. The function $\mu(\theta)$ is continuous in $[0,+\infty)$.

With a small function $\mu(\theta)$ system (1) is close to the system with slowly varying coefficients

$$\frac{dx_0}{dt}=A_0(t,\varepsilon)x_0.$$

The terms depending on θ in system (1) has the order $O(\mu)$. We study the problem of reducing system (1) to the form where the terms depending on θ has the order $O(\mu^{r+1})$, or $O(\varepsilon)$. If a parameter ε is sufficiently small, then the transformed system will be closer to a system with slowly varying coefficients than to system (1).

Theorem. *Let system* (1) *satisfy the following conditions:*

1) *eigenvalues* $\lambda_j(t,\varepsilon)$ $(j=\overline{1,N})$ *of matrix* $A_0(t,\varepsilon)$ *are such that*

$$\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)=in_{jk}\varphi(t,\varepsilon),\ \ n_{jk}\in\mathbf{Z},$$

where the function $\varphi(t,\varepsilon)$ *are defined by condition* (2)*;*

2) *there exists a matrix* $L(t,\varepsilon)$*, the elements of which belong to the class* $S(m)$ *such that* $\inf\limits_G|\det L(t,\varepsilon)|>0$*, and*

$$L^{-1}(t,\varepsilon)A_0(t,\varepsilon)L(t,\varepsilon)=\Lambda(t,\varepsilon)=\operatorname{diag}\big[\lambda_1(t,\varepsilon),\dots,\lambda_N(t,\varepsilon)\big];$$

3) *the function* $\mu(\theta)$ *is such that*

$$\mu(\theta)\in\mathbf{R},\ \ \sup_{[\mathbf{0},+\infty)}\mu(\theta)\le\mu_{\mathbf{0}}<+\infty,\ \ \int\limits_{\mathbf{0}}^{+\infty}\mu^{\mathbf{k}}(\theta)\,\mathbf{d}\theta\le\mu_{\mathbf{0}}<+\infty\ \ (\mathbf{k}=\overline{\mathbf{1},\mathbf{r}}).$$

Then for sufficiently small values of μ_0 *there exists the transformation of the kind*

$$x=\Phi(t,\varepsilon,\theta(t,\varepsilon))y,$$

where the elements of the matrix $\Phi(t,\varepsilon,\theta(t,\varepsilon))$ *are bounded on* $G\times[t_0,+\infty)$*, that leads system* (1) *to the kind*

$$\frac{dy}{dx}=\big(\Lambda(t,\varepsilon)+\varepsilon V(t,\varepsilon,\theta)+W(t,\varepsilon,\theta)\big)y,\tag{3}$$

where the elements of the matrices $V(t,\varepsilon,\theta)$ *and* $W(t,\varepsilon,\theta)$ *are bounded on* $G\times[t_0,+\infty)$*, and the elements of the matrix* $W(t,\varepsilon,\theta)$ *has the order* μ_0^{r+1}*.*

Proof. We make in system (1) the substitution

$$x=L(t,\varepsilon)x^{(1)},$$

where $x^{(1)}$ – new unknown vector od dimension N. We obtain

$$\frac{dx^{(1)}}{dt}=\Big(\Lambda(t,\varepsilon)+\varepsilon H(t,\varepsilon)+\sum_{s=1}^{r}B_s(t,\varepsilon,\theta)(\mu(\theta))^s\Big)x^{(1)},\tag{4}$$

where

$$H(t,\varepsilon)=-\frac{1}{\varepsilon}L^{-1}(t,\varepsilon)\frac{dL(t,\varepsilon)}{dt}\,,\quad B_s(t,\varepsilon,\theta)=L^{-1}(t,\varepsilon)A_s(t,\varepsilon,\theta)L(t,\varepsilon).\tag{5}$$

The elements of the matrix $H(t,\varepsilon)$ belong to the class $S(m-1)$.

We seek the transformation, which leads system (4) to the kind (3), in the form

$$\frac{dx^{(1)}}{dt}=\Big(E+\sum_{s+1}^{r}Q_s(t,\varepsilon,\theta)\Big)y,\tag{6}$$

where the matrices $Q_s(t,\varepsilon,\theta)$ $(s=\overline{1,r})$ are defined from the next chain of the differential equations

$$\varphi(t,\varepsilon)\,\frac{\partial Q_1}{\partial\theta}=\Lambda(t,\varepsilon)Q_1-Q_1\Lambda(t,\varepsilon)+B_1(t,\varepsilon,\theta)\mu(\theta),\tag{7}$$

$$\varphi(t,\varepsilon)\,\frac{\partial Q_2}{\partial\theta}=\Lambda(t,\varepsilon)Q_2-Q_2\Lambda(t,\varepsilon)+B_2(t,\varepsilon,\theta)(\mu(\theta))^2+B_1(t,\varepsilon,\theta)Q_1t,\varepsilon,\theta)\mu(\theta),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\varphi(t,\varepsilon)\,\frac{\partial Q_r}{\partial\theta}=\Lambda(t,\varepsilon)Q_r-Q_r\Lambda(t,\varepsilon)+B_r(t,\varepsilon,\theta)(\mu(\theta))^r+\sum_{s=1}^{r-1}B_s(t,\varepsilon,\theta)Q_{r-s}(t,\varepsilon,\theta)(\mu(\theta))^s.$$

The matrices $V(t,\varepsilon,\theta)$, $W(t,\varepsilon,\theta)$ are defined from the equations

$$\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)V=H(t,\varepsilon)\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)-\frac{1}{\varepsilon}\sum_{s=1}^{r}\frac{\partial Q_s(t,\varepsilon,\theta)}{dt}\,, \tag{8}$$

$$\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)W=\sum_{j=1}^{r}\sum_{s=j}^{r}B_s(t,\varepsilon,\theta)Q_{r+j-s}(t,\varepsilon,\theta)(\mu(\theta))^s. \tag{9}$$

Let

$$Q_s=(q_{jk}^{(s)})_{j,k=\overline{1,N}},\quad B_s=(b_{jk}^{(s)})_{j,k=\overline{1,N}},\ \ s=\overline{1,r}.$$

Consider equation (7). By virtue condition 1) of the theorem equation (7) is equal to the set of scalar equations

$$\frac{\partial q_{jk}^{(1)}}{\partial\theta}=in_{jk}q_{jk}^{(1)}+\frac{1}{\varphi(t,\varepsilon)}\,\mu(\theta)b_{jk}^{(1)}(t,\varepsilon,\theta),\ \ j,k=\overline{1,N}. \tag{10}$$

For each of equations (10), we consider its solution

$$q_{jk}^{(1)}(t,\varepsilon,\theta)=\frac{1}{\varphi(t,\varepsilon)}\,e^{in_{jk}\theta}\int\limits_0^\theta\mu(\vartheta)b_{jk}^{(1)}(t,\varepsilon,\vartheta)e^{-in_{jk}\vartheta}d\vartheta,\ \ j,k=\overline{1,N}. \tag{11}$$

From the fact that elements of matrices $A_s(t,\varepsilon,\theta)$ in system (1) belong to the class $S(m)$ with respect to t,ε, and are continuous and 2π-periodic with respect to $\theta\in[0,+\infty)$, and from equality (5) it follows that the elements of the matrices $B_s(t,\varepsilon,\theta)$ also have similar properties. Hence

$$\sup_{G\times[0,+\infty)}|b_{jk}^{(1)}(t,\varepsilon,\theta)|=c_{jk}^{(1)}<+\infty,\ \ j,k=\overline{1,N}.$$

From (11) and condition 3) of the theorem we have

$$\sup_{G\times[0,+\infty)}\big|q_{jk}^{(1)}(t,\varepsilon,\theta)\big|\le\frac{1}{\varphi_0}\,\mu_0c_{jk}^{(1)},\ \ j,k=\overline{1,N}.$$

For $q_{jk}^{(r)}(t,\varepsilon,\theta)$ we define

$$\begin{aligned}q_{jk}^{(r)}(t,\varepsilon,\theta)&=\frac{1}{\varphi(t,\varepsilon)}\,e^{in_{jk}\theta}\\&\times\int\limits_0^\theta\Big((\mu(\vartheta))^rb_{jk}^{(r)}(t,\varepsilon,\vartheta)+\sum_{s=1}^{r-1}(\mu(\vartheta))^s\sum_{l=1}^{N}b_{jl}^{(s)}(t,\varepsilon,\vartheta)q_{lk}^{(r-k)}(t,\varepsilon,\vartheta)\Big)e^{-in_{jk}\vartheta}\,d\vartheta,\ \ j,k=\overline{1,N}.\end{aligned}$$

All functions $q_{jk}^{(s)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N},\ s=\overline{1,r-1})$ are bounded at $t\in G\times[t_0,+\infty)$. All functions $b_{jk}^{(s)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N},\ s=\overline{1,r})$ are bounded also at $t\in G\times[t_0,+\infty)$. Hence, the condition 3) of the theorem guarantees existence of bounded solutions $q_{jk}^{(r)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N})$, and these solutions have the order μ_0^r. For the small μ_0 the same condition guarantees non-degeneracy of transformation (6). The matrix $V(t,\varepsilon,\theta)$ is uniquely defined from equation (8), and the matrix $W(t,\varepsilon,\theta)$ is uniquely defined from equation (9), and how easy it is to see that the order of the elements of the matrix $W(t,\varepsilon,\theta)$ are not less than μ_0^{r+1}. $\square$

Asymptotic Representations for Solutions of Non-Linear Systems of Ordinary Differential Equations

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We consider the system of differential equations

$$y_i' = f_i(t, y_1, \dots, y_n), \quad i = \overline{1, n}, \tag{1}$$

where $f_i : [a, \omega[\times \prod\limits_{i=1}^{n} \Delta(Y_i^0) \to \mathbb{R}$, $i = \overline{1, n}$, are continuous functions, $-\infty < a < \omega \le +\infty$[1], $\Delta(Y_i^0)$, $i \in \{1, \dots, n\}$ is one-sided neighborhood of Y_i^0, Y_i^0 equals either 0 or $\pm\infty$.

Definition 1. A solution $(y_i)_{i=1}^n$ of system (1) is called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$y_i(t) \in \Delta(Y_i^0) \ \text{ while } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0, \tag{2}$$

$$\lim_{t \uparrow \omega} \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} = \Lambda_i \ \ (i = \overline{1, n-1}). \tag{3}$$

System (1) was considered in T. A. Chanturia's works [1, 2]. In these works, T. A. Chanturia obtained results about existence of proper, singular and oscillating solutions of system (1). These results are especially effective for cyclic systems.

In [3–5, 7, 8], the asymptotics for $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions for cyclic differential equations systems of the following form were considered

$$y_i' = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) \ \ (i = \overline{1, n})^2,$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1, n})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1, n})$ are continuous functions, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = \overline{1, n})$ are continuously differentiable functions and satisfy conditions

$$\lim_{\substack{y_i \to Y_i^0 \\ y_i \in \Delta(Y_i^0)}} \frac{y_i \varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \ \ (i = \overline{1, n}), \quad \prod_{i=1}^{n} \sigma_i \ne 1.$$

Assume that the definition of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution does not give the direct connection between the first and the $n-$th components of this solution. In order to establish this connection, we define the following functions

$$\lambda_i(t) = \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} \ \ (i = \overline{1, n}).$$

[1]For $\omega = +\infty$ consider $a > 0$.

[2]Here and in the sequel, all functions and parametres with the subscript $n+1$ are assumed to coincide with those with the subscript 1.

We proceed and show that

$$\lambda_n(t)=\frac{y_n(t)y_1'(t)}{y_n'(t)y_1(t)}=\frac{y_n(t)y_{n-1}'(t)}{y_n'(t)y_{n-1}(t)}\cdot\frac{y_{n-1}(t)y_{n-2}'(t)}{y_{n-1}'(t)y_{n-2}(t)}\cdots\frac{y_2(t)y_1'(t)}{y_2'(t)y_1(t)}=\frac{1}{\lambda_1(t)\cdots\lambda_{n-1}(t)}\,. \tag{4}$$

From (3) it follows that $\lim\limits_{t\uparrow\omega}\lambda_i(t)=\Lambda_i$ $(i=\overline{1,n-1})$. Therefore, if there are zeros among Λ_i $(i=\overline{1,n-1})$ from (4), we obtain

$$\Lambda_n=\lim_{t\uparrow\omega}\lambda_n(t)=\pm\infty.$$

In particular, it is evident that the case, when among all Λ_i $(i=1,\dots,n-1)$ there is a single $\pm\infty$, while all others are real numbers different from zero, could be transformed into the case described in this work. This transformation is carried out by cyclic redesignation of variables, functions and constants. For instance, if $\Lambda_l=\pm\infty$ $(l\in\{1,\dots,n-1\})$, the indices are redesignated as follows

$$l\longrightarrow n,\ l+1\longrightarrow 1,\ \dots,\ n\longrightarrow n-l,\ 1\longrightarrow n-l+1,\ \dots,\ l-1\longrightarrow n-1.$$

It is obvious that $\Lambda_i=0$ when $i=n-l$.

Further, we introduce auxiliary notation.

First, if

$$\mu_i=\begin{cases}1 & \text{as } Y_i^0=+\infty, \text{ or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of } 0,\\ -1 & \text{as } Y_i^0=-\infty, \text{ or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is left neighborhood of } 0,\end{cases}$$

it is obvious that μ_i $(i=\overline{1,n})$ determine the signs of the components of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution in some left neighborhood of ω.

The existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions of system (1) for fixed values of $\Lambda_i\in\mathbb{R}$ $(\prod\limits_{i=1}^{n-1}\Lambda_i=0)$, $i=\overline{1,n-1}$, and their asymptotics as $t\uparrow\omega$ will be explored when this system is in a certain way close to a cyclic one with regularly varying non-linearities.

Definition 2. We say that system (1) satisfies the condition $N(\Lambda_1,\dots,\Lambda_{n-1})$, where $\Lambda_i\in\mathbb{R}$, $i=\overline{1,n-1}$, if for any $k\in\{1,\dots,n\}$ there exist a number $\alpha_k\in\{-1,1\}$, a continuous function $p_k:[a,\omega[\,\to]0,+\infty[$ and continuous regularly varying $\varphi_{k+1}:\Delta(Y_{k+1}^0)\to]0;+\infty[$ of σ_{k+1} orders (when $y_{k+1}\to Y_{k+1}^0$) which admit the following representation for any functions $y_i:[a,\omega[\,\to\Delta(Y_i^0)$, $i=\overline{1,n}$, satisfying conditions (2), (3):

$$f_k(t,y_1(t),\dots,y_n(t))=\alpha_k p_k(t)\varphi_{k+1}(y_{k+1}(t))[1+o(1)] \ \text{ when } \ t\uparrow\omega. \tag{5}$$

Since functions φ_i $(i=\overline{1,n})$ are regularly varying as $z\to Y_i^0$ of σ_i orders, they admit the following representation (see [6]):

$$\varphi_i(y_i)=|y_i|^{\sigma_i}\theta_i(y_i)\ \ (i=\overline{1,n}), \tag{6}$$

where $\theta_i:\Delta(Y_i^0)\to]0;+\infty[$ $(i=\overline{1,n})$ are slowly varying functions as $z\to Y_i^0$.

Having supposed that system (1) for certain Λ_i, $i\in\{1,\dots,n-1\}$, satisfies the condition $N(\Lambda_1,\dots,\Lambda_{n-1})$ and $\prod\limits_{k=1}^{n}\sigma_k\neq 1$ (for orders σ_k, $k=\overline{1,n}$ of functions φ_k), we introduce auxiliary designation.

We denote sets

$$\mathfrak{I}=\big\{i\in\{1,\dots,n-1\}:\ 1-\Lambda_i\sigma_{i+1}\neq 0\big\},\quad \overline{\mathfrak{I}}=\{1,\dots,n-1\}\setminus\mathfrak{I}$$

and suppose that $1-\Lambda_{n-1}\sigma_n \neq 0$.

By taking into account the fact that $n-1 \in \mathfrak{I}$, we denote auxiliary functions I_i, Q_i $(i=1,\dots,n)$ and non-zero constants β_i $(i=1,\dots,n)$, supposing that

$$I_i(t)=\begin{cases} \displaystyle\int\limits_{A_i}^{t} p_i(\tau)\,d\tau & \text{for } \ i\in\mathfrak{I},\\ \displaystyle\int\limits_{A_i}^{t} p_i(\tau)I_{i+1}(\tau)\,d\tau & \text{for } \ i\in\overline{\mathfrak{I}},\\ \displaystyle\int\limits_{A_n}^{t} p_n(\tau)q_n(\tau)\,d\tau & \text{for } \ i=n, \end{cases}$$

$$\beta_i=\begin{cases} 1-\Lambda_i\sigma_{i+1} & \text{if } \ i\in\mathfrak{I},\\ \beta_{i+1}\Lambda_i & \text{if } \ i\in\overline{\mathfrak{I}},\\ 1-\displaystyle\prod_{k=1}^{n}\sigma_k & \text{if } \ i=n, \end{cases} \qquad Q_i(t)=\begin{cases} \alpha_i\beta_i I_i(t) & \text{for } \ i\in\mathfrak{I}\cup\{n\},\\ \dfrac{\alpha_i\beta_i I_i(t)}{I_{i+1}(t)} & \text{for } \ i\in\overline{\mathfrak{I}}, \end{cases}$$

where each limit of integration $A_i\in\{\omega,a\}$ $(i\in\{1,\dots,n-1\})$, $A_n\in\{\omega,b\}$ ($b\in[a,\omega[$) is chosen in such a way that its corresponding integral I_i aims either to zero, or to ∞ as $t\uparrow\omega$,

$$q_n(t)=\theta_1\big(\mu_1|I_1(t)|^{\frac{1}{\beta_1}}\big)|Q_{n-1}(t)|^{\prod\limits_{k=1}^{n-1}\sigma_k}\prod_{k=1}^{n-2}\Big|Q_k(t)\theta_{k+1}\big(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\big)\Big|^{\prod\limits_{i=1}^{k}\sigma_i}.$$

In addition, we introduce numbers

$$A_i^*=\begin{cases} 1 & \text{if } \ A_i=a,\\ -1 & \text{if } \ A_i=\omega \end{cases} \quad (i=1,\dots,n-1), \quad A_n^*=\begin{cases} 1 & \text{if } \ A_n=b,\\ -1 & \text{if } \ A_n=\omega. \end{cases}$$

These numbers enable us to define the signs of functions I_i $(i=1,\dots,n-1)$ on the interval $]a,\omega[$ and the sign of function I_n on the interval $]b,\omega[$.

Definition 3. We say that the function φ_k $(k\in\{1,\dots,n\})$ satisfies the condition $\mathbf{S}$ if for any continuously differentiable function $l:\Delta(Y_k^0)\to\,]0,+\infty[$ with the property

$$\lim_{\substack{z\to Y_k^0\\ z\in\Delta(Y_k^0)}}\frac{z\,l'(z)}{l(z)}=0,$$

the function θ_k (defined in (6)) admits the asymptotic representation

$$\theta_k(zl(z))=\theta_k(z)[1+o(1)] \ \text{ when } \ z\to Y_k^0 \ \ (z\in\Delta(Y_k^0)).$$

For instance, $\mathbf{S}$ – condition is obviously satisfied by functions φ_k of the following type

$$\varphi_k(y_k)=|y_k|^{\sigma_k}|\ln y_k|^{\gamma_1}, \quad \varphi_k(y_k)=|y_k|^{\sigma_k}|\ln y_k|^{\gamma_1}|\ln|\ln y_k||^{\gamma_2},$$

where $\gamma_1,\gamma_2\neq 0$. $\mathbf{S}$ – condition is also satisfied by functions φ_k which include functions θ_k that have the eventual limit as $y_k\to Y_k^0$. $\mathbf{S}$ – condition is also satisfied by many other functions.

By means of introduced designations, we will establish the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions for (1).

Theorem. *Let system* (1) *satisfy* $N(\Lambda_1,\dots,\Lambda_{n-1})$*-condition and* $\Lambda_i \in \mathbb{R}$ $(i=\overline{1,n-1})$ *include those equal zeros,* $n-1 \in \mathfrak{I}$ *and* $m=\max\{i \in \mathfrak{I}:\ \Lambda_i=0\}$. *Let also functions* φ_k $(k=\overline{1,n-1})$, *defined in* (5), *satisfy* $\mathbf{S}$*-condition. Then for the existence of* $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$*-solutions of system* (1) *it is necessary, and if the algebraic equation*

$$(1+\lambda)\prod_{j=m+1}^{n-1}(M_j+\lambda)=\frac{\prod\limits_{j=1}^{n}\sigma_j}{\prod\limits_{j=1}^{n}\sigma_j-1}\Big(\sum_{k=m}^{n-1}\prod_{j=m+1}^{k}(M_j+\lambda)\prod_{s=k+2}^{n-1}M_s\Big)\lambda\,,^3 \tag{7}$$

where

$$M_j=\Big(\prod_{i=j}^{n-1}\Lambda_i\Big)^{-1}\ \ (j=\overline{m+1,n-1}),$$

does not have roots with zero real part, it is also sufficient that

$$\lim_{t\uparrow\omega}\frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)}=\Lambda_i\frac{\beta_{i+1}}{\beta_i}\ \ (i=\overline{1,n-1}),$$

and for each $i\in\{1,\dots,n\}$ *the following conditions are satisfied*

$$A_i^*\beta_i>0\ \text{ if } Y_i^0=\pm\infty,\quad A_i^*\beta_i<0\ \text{ if } Y_i^0=0,$$
$$\operatorname{sign}[\alpha_iA_i^*\beta_i]=\mu_i.$$

Moreover, components of each solution of that type admit the following asymptotic representation as $t\uparrow\omega$

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))}=Q_i(t)[1+o(1)]\ \ (i=\overline{1,n-1}),$$
$$\frac{y_n(t)}{[\varphi_n(y_n(t))]^{\prod\limits_{i=1}^{n-1}\sigma_i}}=Q_n(t)[1+o(1)],$$

and there exists the whole k*-parametric family of these solutions if there are* k *positive roots among the solutions of the following algebraic equation*

$$\gamma_i=\begin{cases}\beta_iA_i^* & \text{if } i\in\mathfrak{I}\setminus\{m+1,\dots,n-1\},\\ \beta_iA_i^*A_{i+1}^* & \text{if } i\in\overline{\mathfrak{I}}\setminus\{m+1,\dots,n-1\},\\ A_n^*\Big(\prod\limits_{j=1}^{n-1}\sigma_j-1\Big)\operatorname{Re}\lambda_{i-m}^0 & \text{if } i\in\{m+1,\dots,n\},\end{cases}$$

where λ_j^0 $(j=\overline{1,n-m})$ *are roots of the algebraic equation* (7) *(along with multiple ones).*

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Initial Value Method in Boundary Value Problems for Systems of Two-Term Fractional Differential Equations at Resonance

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$ and $X = C(J) \times C(J)$.

We investigate the system of fractional differential equations

$$\left.\begin{aligned} {}^c\!D^{\alpha} u(t) + p(t)\,{}^c\!D^{\alpha_1} u(t) = f(t, u(t), v(t)),\\ {}^c\!D^{\beta} v(t) + q(t)\,{}^c\!D^{\beta_1} v(t) = g(t, u(t), v(t)), \end{aligned}\right\} \tag{1.1}$$

where $0 < \alpha_1 < \alpha \le 1$, $0 < \beta_1 < \beta \le 1$, $p, q \in C(J)$, $f, g \in C(J \times \mathbb{R}^2)$ and ${}^c\!D$ denotes the Caputo fractional derivative.

Let $\mathcal{K}, \mathcal{R} : C(J) \to \mathbb{R}$ be functionals given as

$$\mathcal{K}x = \sum_{k=1}^{m_1} c_k x(\rho_k), \qquad \mathcal{R}x = \sum_{k=1}^{m_2} d_k x(\xi_k),$$

where $m_j \in \mathbb{N}$ or $m_j = \infty$, $j = 1, 2$, $\{\rho_k\}_{k=1}^{m_1} \subset (0, T]$, $\{\xi_k\}_{k=1}^{m_2} \subset (0, T]$ are increasing sequences and $c_k > 0$, $d_k > 0$, $\sum\limits_{k=1}^{m_1} c_k = 1$, $\sum\limits_{k=1}^{m_2} d_k = 1$.

Together with system (1.1) we study the boundary condition

$$(u(0), v(0)) = (\mathcal{K}u, \mathcal{R}v). \tag{1.2}$$

Definition 1.1. We say that $(u, v) : J \to \mathbb{R}^2$ is *a solution of system* (1.1) if $(u, v), ({}^c\!D^{\alpha} u, {}^c\!D^{\beta} v) \in X$ and (u, v) satisfies (1.1) for $t \in J$. A solution (u, v) of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

Since each constant vector-function (u, v) on the interval J is a solution of problem ${}^c\!D^{\alpha} u + p(t)\,{}^c\!D^{\alpha_1} u = 0$, ${}^c\!D^{\beta} v + q(t)\,{}^c\!D^{\beta_1} v = 0$, (1.2), problem (1.1), (1.2) is at resonance.

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [1, 2].

The Riemann–Liouville fractional integral $I^{\gamma} x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma} x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\, \mathrm{d}s,$$

where *Gamma* is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c D^\gamma x$ of order $\gamma \in (0,1)$ of a function $x : J \to \mathbb{R}$ is given as

$$ {}^c D^\gamma x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\, (x(s) - x(0))\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\, I^{1-\gamma}\, (x(t) - x(0)). $$

If $\gamma = 1$, then ${}^c D^\gamma x(t) = x'(t)$.

The special case of (1.1) (for $\alpha = 1$, $\beta = 1$) is the system of generalized Basset fractional differential equations [3]

$$ \left. \begin{aligned} u'(t) + p(t)\,{}^c D^{\alpha_1} u(t) = f(t, u(t), v(t)), \\ v'(t) + q(t)\,{}^c D^{\beta_1} v(t) = g(t, u(t), v(t)). \end{aligned} \right\} $$

The special cases of (1.2) are the periodic condition

$$ (u(0), v(0)) = (u(T), v(T)) $$

and the infinite-point boundary condition

$$ (u(0), v(0)) = \Big(\sum_{k=1}^\infty c_k u(\rho_k), \sum_{k=1}^\infty d_k v(\xi_k) \Big). $$

We will work with the following conditions for the functions p, q, f and g in (1.1):

(H_1) There exist $D, H, K, L \in \mathbb{R}$, $D < H$, $K < L$, such that

$$ \begin{aligned} f(t, D, y) > 0, \ \ f(t, H, y) < 0 \ \text{ for } t \in J,\ y \in [K, L], \\ g(t, x, K) > 0, \ \ f(t, x, L) < 0 \ \text{ for } t \in J,\ x \in [D, H]. \end{aligned} $$

(H_2) $p(t) \ge 0$ and $q(t) \ge 0$ for $t \in J$.

The aim of this paper is to discuss the existence of solutions to problem (1.1), (1.2). The existence results are proved by the following procedure. By the combination of initial value method [4] with the maximum principle for the Caputo fractional derivative [4] and the Schaefer fixed point theorem we first prove that for each $(c_1, c_2) \in [D, H] \times [K, L]$ there exists a solution (u, v) of system (1.1) on the interval J satisfying the initial condition $(u(0), v(0)) = (c_1, c_2)$. Then we discuss the set $\mathcal{C}$ of all such solutions and show that $\mathcal{C}$ is a compact metric space. Assuming that $(u(0), v(0)) \neq (\mathcal{K}u, \mathcal{R}v)$ for all $(u, v) \in \mathcal{C}$ we obtain a contradiction by the study of some compact subsets of $\mathcal{C}$.

2 Initial value problem

For $r \in C(J)$ and $\gamma \in (0,1)$, let $\Lambda_{r,\gamma} : C(J) \to C(J)$ be defined as

$$ \Lambda_{r,\gamma} x(t) = -r(t) I^\gamma x(t) $$

and $\Lambda^0_{r,\gamma}$ be the identical operator on $C(J)$. For $n \in \mathbb{N}$, let $\Lambda^n_{r,\gamma} = \underbrace{\Lambda_{r,\gamma} \circ \Lambda_{r,\gamma} \circ \cdots \circ \Lambda_{r,\gamma}}_{n}$ be nth iteration of $\Lambda_{r,\gamma}$. Let $\mathcal{D}_{r,\gamma} : C(J) \to C(J)$ be an operator defined as

$$ \mathcal{D}_{r,\gamma} x(t) = \sum_{n=0}^\infty \Lambda^n_{r,\gamma} x(t). $$

Let (H_1) hold. Let

$$\eta(x)=\begin{cases}H & \text{if } x>H,\\ x & \text{if } x\in[D,H],\\ D & \text{if } x<D,\end{cases}\qquad \rho(y)=\begin{cases}L & \text{if } y>L,\\ y & \text{if } y\in[K,L],\\ K & \text{if } y<K,\end{cases}$$

and $f^*,g^*:J\times\mathbb{R}^2\to\mathbb{R}$ be given as

$$f^*(t,x,y)=f(t,\eta(x),\rho(y)),\quad g^*(t,x,y)=g(t,\eta(x),\rho(y)).$$

Then $f^*,g^*\in C(J\times\mathbb{R}^2)$ are bounded and

$$\left.\begin{array}{ll} f^*(t,x,y)>0 \text{ if } x<D,\ y\in\mathbb{R}, & f^*(t,x,y)<0 \text{ if } x>H,\ y\in\mathbb{R},\\ g^*(t,x,y)>0 \text{ if } x\in\mathbb{R},\ y<K, & g^*(t,x,y)<0 \text{ if } x\in\mathbb{R},\ y>L,\end{array}\right\}$$

for $t\in J$. Let operators $\mathcal{F},\mathcal{G}:X\to C(J)$ be the Nemytskii operators associated to f^*, g^*,

$$\mathcal{F}(x,y)(t)=f^*(t,x(t),y(t)),\quad \mathcal{G}(x,y)(t)=g^*(t,x(t),y(t)),$$

and $\mathcal{A},\mathcal{B}:C(J)\to C(J)$,

$$\mathcal{A}x(t)=\mathcal{D}_{p,\alpha-\alpha_1}x(t),\quad \mathcal{B}x(t)=\mathcal{D}_{q,\beta-\beta_1}x(t),$$

where p, q, α, α_1, β and β_1 are from (1.1).

We now consider the fractional initial value problem

$$\left.\begin{aligned} {}^cD^{\alpha}u(t)+p(t){}^cD^{\alpha_1}u(t)&=f^*(t,u(t),v(t)),\\ {}^cD^{\beta}v(t)+q(t){}^cD^{\beta_1}v(t)&=g^*(t,u(t),v(t)),\end{aligned}\right\}\tag{2.1}$$

$$(u(0),v(0))=(c_1,c_2),\ \ (c_1,c_2)\in\mathbb{R}^2.\tag{2.2}$$

Let an operator $\mathcal{Q}:X\to X$ be defined by the formula

$$\mathcal{Q}(x,y)=(\mathcal{Q}_1(x,y),\mathcal{Q}_2(x,y)),$$

where $Q_j:X\to C(J)$,

$$\mathcal{Q}_1(x,y)(t)=c_1+I^{\alpha}\mathcal{A}\mathcal{F}(x,y)(t),\quad \mathcal{Q}_2(x,y)(t)=c_2+I^{\beta}\mathcal{B}\mathcal{G}(x,y)(t),$$

and c_1, c_2 are from (2.2).

The following result gives the relation between solutions of problem (2.1), (2.2) and fixed points of $\mathcal{Q}$.

Lemma 2.1. *Let (H_1) hold. Then (u,v) is a fixed point of $\mathcal{Q}$ if and only if (u,v) is a solution of problem* (2.1), (2.2).

The existence results for problems (2.1), (2.2) and (1.1), (2.2) are stated in the following two lemmas.

Lemma 2.2. *Let (H_1) hold. Then there exists at least one solution of problem* (2.1), (2.2).

Let $\Delta=[D,H]\times[K,L]$, where D,H,K and L are from (H_2).

Lemma 2.3. *Let (H_1), (H_2) hold and let $(c_1,c_2)\in\Delta$. Then problem* (1.1), (2.2) *has at least one solution and all its solutions (u,v) satisfy*

$$D<u(t)<H,\ \ K<v(t)<L\ \ \textit{for}\ \ t\in(0,T].\tag{2.3}$$

3 Existence result for problem (1.1), (1.2)

Theorem 3.1. *Let (H_1) and (H_2) hold. The problem* (1.1), (1.2) *has at least one solution (u,v) and*

$$D < u(t) < H, \ \ K < v(t) < L \ \ \text{for} \ \ t \in J. \tag{3.1}$$

Sketch of proof. Having in mind Lemma 2.3, for $(c_1,c_2) \in \Delta$ let $\mathcal{C}_{(c_1,c_2)}$ be the set of all solutions to problem (1.1), (2.2). Let

$$\mathcal{C} = \bigcup_{(c_1,c_2)\in\Delta} \mathcal{C}_{(c_1,c_2)}.$$

Then for each $(u,v) \in \mathcal{C}$ the equalities

$$u(t) = u(0) + I^{\alpha}\mathcal{A}\mathcal{F}(u,v)(t), \quad v(t) = v(0) + I^{\beta}\mathcal{B}\mathcal{G}(u,v)(t), \ \ t \in J,$$

and inequality (2.3) hold. We can prove that $\mathcal{C}$ is a compact metric space equipped with the metric

$$\rho((u,v),(u_1,v_1)) = \max\big\{|u(t)-u_1(t)| : \ t \in J\big\} + \max\big\{|v(t)-v_1(t)| : \ t \in J\big\}.$$

Assume to the contrary that

$$|u(0)-\mathcal{K}u| + |v(0)-\mathcal{R}v| > 0 \ \ \text{for} \ \ (u,v) \in \mathcal{C}, \tag{3.2}$$

where $\mathcal{K},\mathcal{R}$ are from the boundary condition (1.2). Condition (3.2) is equivalent to

$$(u,v) \in \mathcal{C} \implies \begin{cases} \text{either } u(0)-\mathcal{K}u = 0 \text{ and } v(0)-\mathcal{R}v \neq 0 \\ \text{or } u(0)-\mathcal{K}u \neq 0 \text{ and } v(0)-\mathcal{R}v = 0. \end{cases} \tag{3.3}$$

Keeping in mind (3.3), let

$$\begin{aligned} \mathcal{P}_1 &= \big\{(u,v) \in \mathcal{C} : \ u(0) = \mathcal{K}u, \ v(0)-\mathcal{R}v \neq 0\big\}, \\ \mathcal{P}_2 &= \big\{(u,v) \in \mathcal{C} : \ u(0)-\mathcal{K}u \neq 0, \ v(0) = \mathcal{R}v\big\}. \end{aligned}$$

Then $\mathcal{C} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$ and we can prove that $\mathcal{P}_1$, $\mathcal{P}_2$ are nonvoid compact subsets of $\mathcal{C}$. Hence the compact metric space $\mathcal{C}$ is the union of nonvoid, mutually disjoint compact subsets $\mathcal{P}_1,\mathcal{P}_2$, which is impossible. As a result assumption (3.2) is false. Consequently, problem (1.1), (1.2) has a solution (u,v).

It remains to prove that (u,v) satisfies inequality (3.1). We know that (u,v) satisfies inequality (2.3). Assume, for example, that $v(0) = K$. Since $v > K$ on $(0,T]$, we have

$$v(0)-\mathcal{R}v = v(0) - \sum_{k=1}^{m_2} d_j v(\xi_j) < K - K\sum_{k=1}^{m_2} d_j = K - K = 0,$$

which contradicts $v(0)-\mathcal{R}v = 0$. Hence $v > K$ on J. □

Example 3.1. Let $r,l,p,q \in C(J)$, $r > 1$, $l > 0$, and let $\rho \geq 1$. Then the functions $f(t,x,y) = r(t) - e^x + e^{-y}$, $g(t,x,y) = l(t) + x - |y|^{\rho}$ satisfy condition (H_1) for $D = 0$, $H = \ln(2+\|r\|)$, $K = 0$ and $L = \sqrt[q]{1+\|l\|+\ln(2+\|r\|)}$. Applying Theorem 3.1, the system

$$\left.\begin{aligned} {}^{c}\!D^{\alpha}u + |p(t)|\,{}^{c}\!D^{\alpha_1}u &= r(t) - e^{u} + e^{-v}, \\ {}^{c}\!D^{\beta}v + |q(t)|\,{}^{c}\!D^{\beta_1}v &= l(t) + u - |v|^{\rho} \end{aligned}\right\}$$

has a solution (u,v) satisfying the boundary condition (1.2) and

$$0 < u(t) < \ln(2+\|r\|), \quad 0 < v(t) < \sqrt[q]{1+\|l\|+\ln(2+\|r\|)}\,, \ \ t \in J.$$

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Application of the Averaging Method to Optimal Control Problems of Systems with Impulse Action in Non-Fixed Moments of Times

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We study the application of the method of averaging to the problems of optimal control over impulsive differential equations. The procedure of averaging allows to replace the original problem with the problem of optimal control by a system of ordinary differential equations. The optimal control problems are investigated on finite and infinite horizons.

Introduction

For a system of differential equations with an impulsed action at non-fixed moments of time

$$\begin{gathered}\dot{x} = \varepsilon X(t,x,u), \quad t \neq t_i(x),\\ \triangle x\big|_{t=t_i(x)} = \varepsilon I_i(x,v_i),\\ x(0)=x_0, \quad t_i(x)<t_{i+1}(x),\end{gathered} \tag{0.1}$$

two optimal control problems on a finite and infinite interval with a quality criterion are considered:

(1) on a finite interval with a quality criterion are considered:

$$J^1_\varepsilon(u,v) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} \Phi(t,x(t),u(t))\,dt + \varepsilon \sum_{0<t_i(x)<\frac{T}{\varepsilon}} \Psi_i(x(t_i),v_i) \longrightarrow \inf, \tag{0.2}$$

(2) on an infinite interval with a quality criterion are considered:

$$J^2_\varepsilon(u,v) = \varepsilon \int\limits_0^{\infty} e^{-\gamma t} L(t,x(t))\,dt \longrightarrow \inf. \tag{0.3}$$

Here $T>0$, $\varepsilon>0$, $\gamma>0$ are fixed; $t\geq 0$, $x\in D$ is a domain in the space R^d, $u\in U\subset R^m$, $v_i\in V\subset R^r$, where U and V are the subsets in the spaces R^m and R^r, respectively. We denote by $|\cdot|$ the Euclidean norm of the vector, and by $\|\cdot\|$ we denote the norm of the matrix consistent with the norm of the vector.

Controls of $u=u(t)=(u_1(t),u_2(t),\dots,u_m(t))$ and $v=v_i=(v_{i1},v_{i2},\dots,v_{ir})$ will be considered admissible for problems (0.1)–(0.3) if:

(a1) the function $u(t)$ is measurable and locally integrated at $t \geq 0$;

(a2) $u(t) \in U$, $t \geq 0$;

(a3) for every $u(t)$ there exists a constant $u_0 \in U$ such that $u(t) \to u_0$ for $t \to \infty$ uniformly for all controls, i.e. for arbitrary $\delta > 0$ there exists a constant $T_0 > 0$, independent of $u(t)$, u_0, such that for all $t \geq T_0$ the inequality $|u(t) - u_0| < \delta$ holds;

(a4) for each sequence of vectors v_i there exists $v_0 \in V$ such that $v_i \to v_0$, $i \to \infty$ uniformly for all controls, i.e. for arbitrary $\delta > 0$ there exists a constant N_0, independent of v_i, v_0, such that for all $i \geq N_0$ the inequality $|v_i - v_0| < \delta$ is satisfied;

(a5) condition $|J_\varepsilon(u,v)| < \infty$ holds for functional (0.3).

Note that conditions (a3) and (a4) are obviously satisfied if there exist a function $\varphi(t) \to 0$, and a sequence $\varphi(t) \to 0$, $t \to \infty$ which are independents of $u(t)$ and v_i, respectively, such that $|u(t) - u_0| \leq \varphi(t)$, $|v_i - v_0| < a_i$. Condition (a3) for control, first appeared in M. M. Moiseyev [3], when applying the method of averaging to practical problems. In this monograph, such controls are called asymptotically constant.

We denote the set of admissible controls of problems (0.1), (0.2) and (0.1)–(0.3) by F_1 and F_2, respectively. In this case,

$$J_\varepsilon^1 = \inf_{(u,v)\in F_1} J_\varepsilon^1(u,v)$$

and

$$J_\varepsilon^2 = \inf_{(u,v)\in F_2} J_\varepsilon^2(u,v).$$

Denote by $x_\varepsilon(t,u,v)$ the solution of the Cauchy problem corresponding to the admissible control (u,v). The triple $(x_\varepsilon^*(t,u,v), u_\varepsilon^*, v_\varepsilon^*)$ is optimal for problems (0.1)–(0.3) if $(u_\varepsilon^*, v_\varepsilon^*)$ is an admissible pair and $J_\varepsilon^1(u_\varepsilon^*, v_\varepsilon^*) = J_\varepsilon^1$ for functional (0.2), or $J_\varepsilon^2(u_\varepsilon^*, v_\varepsilon^*) = J_\varepsilon^2$ for functional (0.3).

Let the averaging conditions be satisfied:

(a6) there are limits uniformly across $t \geq 0$, $x \in D$, $u \in U$, $v \in V$:

$$\lim_{s\to\infty} \frac{1}{s} \int_t^{s+t} X(\tau,x,u)\,d\tau = X_0(x,u), \tag{0.4}$$

$$\lim_{s\to\infty} \frac{1}{s} \sum_{t<t_i(x)<s+t} I_i(x,v) = I_0(x,v), \tag{0.5}$$

$$\lim_{s\to\infty} \frac{1}{s} \int_t^{s+t} \Phi(\tau,x,u)\,d\tau = \Phi_0(x,u), \tag{0.6}$$

$$\lim_{s\to\infty} \frac{1}{s} \sum_{t<t_i(x)<s+t} \Psi_i(x,v) = \Psi_0(x,v). \tag{0.7}$$

With respect to the moments of impulse action, we will assume that there exists a constant $C > 0$ such that for $t \geq 0$, $x \in D$

$$\sum_{t<t_i(x)<s+t} I_i \leq Cs. \tag{0.8}$$

We will put averaged tasks in accordance with the problems of optimal control (0.1)–(0.3)

$$\dot{y} = \varepsilon[X_0(y,u_0) + I_0(y,v_0)], \quad y(0) = x_0, \tag{0.9}$$

$$\overline{J}_\varepsilon^{\,1}(u_0,v_0) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} \big[\Phi_0(y(t),u_0) + \Psi_0(y(t),v_0)\big]\,dt \longrightarrow \inf, \tag{0.10}$$

$$\overline{J}_\varepsilon^{\,2}(u_0,v_0) = \varepsilon \int\limits_0^{\infty} e^{-\gamma t} L(t,y(t))\,dt \longrightarrow \inf, \tag{0.11}$$

where $u_0 \in U$, $v_0 \in V$ are already constant vectors. These tasks are much simpler than the original ones because they are problems of optimal control for systems of ordinary differential equations. Denote by analogy as in the case of initial problems $\overline{J}_\varepsilon^{\,1} = \inf\limits_{(u_0,v_0)\in F_1} \overline{J}_\varepsilon^{\,1}(u_0,v_0)$ and $\overline{J}_\varepsilon^{\,2} = inf_{(u,v)\in F_2} \overline{J}_\varepsilon^{\,2}(u_0,v_0)$.

The main result is obtained which states that the optimal control $(u_0^*(\varepsilon), v_0^*(\varepsilon))$ of averaged problems is η-optimal for the initial problems, namely, for arbitrary $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the inequalities:

$$\big|J_\varepsilon^1(u_0^*(\varepsilon), v_0^*(\varepsilon)) - J_\varepsilon^1\big| < \eta, \quad \big|J_\varepsilon^2(u_0^*(\varepsilon), v_0^*(\varepsilon)) - J_\varepsilon^2\big| < \eta$$

are satisfied.

It is known that the averaging method is one of the most common methods of analyzing nonlinear dynamic systems. For ordinary differential equations, this method was substantiated by M. M. Bogolyubovym [1]. The validation of this method for systems with impulse action in the general form was first obtained in [6]. We also note the works [7, 9], where the results of [6] have been further developed.

The averaging method also proved to be effective for solving problems of optimal control. A number of papers are devoted to this question (see, for example, [5], where there is an extensive bibliography). In [4] developed a different approach as for to applying the averaging method to tasks of optimal control, namely, considering the control function u as a parameter, was averaging over by time, that clearly included in the right-parts sides of the system.

In this paper, the approach under consideration is applied to the problems of optimal control of impulse systems with non-fixed moments of impulse actions. Such problems with the application of the principle of maximum were previously studied in [8].

This paper describes the problem formulation and reviews the literature, gives strict formulation of the problem, and presents the main results obtained when solving the problems under consideration.

1 Statement of the problem and formulation of the main results

In what follows, we consider the following conditions for problems (0.1)–(0.3) and their corresponding averaged problems (0.9)–(0.11):

2.1. The functions X, I_i, Φ, Ψ_i, L are uniformly continuous on the set of variables at $t \geq 0$, $x \in D$, $u \in U$, $v \in V$, evenly at $i = 1, 2, \dots$.

2.2. There is a positive constant M such that

$$\Big|\frac{\partial t_i(x)}{\partial x}\Big| + |X(t,x,u)| + |\Phi(t,x,u)| + |\Psi_i(x,v)| + |I_i(x,v)| \leq M,$$

for $t \geq 0$, $x \in D$, $u \in U$, $v \in V$, $i = 1, 2, \dots$.

2.3. There is a positive constant K such that

$$|X(t,x,u)-X(t,x_1,u)|+|I_i(x,v)-I_i(x_1,v)|+|\Phi(t,x,u)-\Phi(t,x_1,u)| \\ +|\Psi_i(x,v)-\Psi_i(x_1,v)|+\Big|\frac{\partial t_i(x)}{\partial x}-\frac{\partial t_i(x_1)}{\partial x}\Big|\le K|x-x_1|, \quad \Big|\frac{\partial t_i(x)}{\partial x}\Big|\le K$$

for $t\ge 0$, $x,x_1\in D$, $i=1,2,\dots$, $u\in U$, $v\in V$.

2.4. Condition (a5) is satisfied.

2.5. The averaged Cauchy problem (0.9) has the solution $y(\varepsilon t)=y(\varepsilon t,x_0,u_0,v_0), y(0,x_0,u_0,v_0)=x_0$, which for $\varepsilon=1$ belongs to D for $t\in[0,T]$ together with some own ρ-circle (independent of u_0, v_0) and the inequality

$$\frac{\partial t_i(y(\varepsilon t))}{\partial x}I_i(y(\varepsilon t),v)\le\beta<0$$

holds when $t_i'<t<t_i''$, $v\in V$, or

$$\frac{\partial t_i(x)}{\partial x}\equiv 0.$$

Here

$$t_i'=\inf_{x\in D}t_i(x), \quad t_i''=\sup_{x\in D}t_i(x), \quad i=\overline{1,l}, \quad t_l<\frac{T}{\varepsilon}<t_{l+1}.$$

The following theorem is on the connection between problems of optimal control on finite time intervals.

Theorem 1.1 ([2])**.** *Let conditions* 2.1–2.5 *be satisfied and there be an optimal control* $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ *of the averaged problem* $(0.9),(0.10)$ *for* $0<\varepsilon\le\varepsilon_0$*. Then for arbitrary* $\eta>0$ *there exists* $\varepsilon_1=\varepsilon_1(\eta,\varepsilon_0)>0$ *such that for all* $\varepsilon\in(0,\varepsilon_1)$ *the following conditions hold:*

(1) $J_\varepsilon^1>-\infty$*;*

(2) *the inequality holds*

$$|J_\varepsilon^1(u_0^*(\varepsilon),v_0^*(\varepsilon))-J_\varepsilon^1|\le\eta. \tag{1.1}$$

Remark 1.1. If the conditions of Theorem 1.1 state that the sets of admissible controls U and V are compact, then the optimal control $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ of the averaged problem exists.

Indeed, the solution of the averaged problem (0.9) extends to the interval $[0,\frac{T}{\varepsilon}]$. Conditions of Theorem 1.1 imply that $y(t,u_0,v_0)$ is a continuous function of the parameters u_0 and v_0, therefore, Lebesgue's theorem on majorized convergence also implies the continuity of $\overline{J}_\varepsilon^1(u_0,v_0)$ over u_0 and v_0. The statement of Remark 1.1 is now a consequence of the Weierstrass theorem.

Remark 1.2. If $X_0(y,u_0)+I_0(y,v_0)$, $\Phi_0(y,u_0)+\Psi_0(y,v_0)$ are continuous differentiated functions, then problem $(0.9),(0.10)$ is a smooth finite-dimensional extremal problem.

Consider the problem of optimal control on the axis, for this system (0.9) we write at "slow time": $\tau=\varepsilon t$:

$$\frac{dy}{d\tau}=\big[X_0(y,u_0)+I_0(y,v_0)\big], \quad y(0)=x_0. \tag{1.2}$$

Theorem 1.2 ([2]). *Let the conditions* 2.1–2.5 *hold, and let the solution* $y(\tau)=y(\tau,x_0,u_0,v_0)$ *of the Cauchy problem* (1.2) *be uniformly asymptotically stable at* τ_0, u_0 *and* v_0, *and belong to the domain* D *at* $\tau\geq 0$ *together with its some* p*-circle (independent of* u_0, v_0*), and the inequalities* $\frac{\partial t_i(x)}{\partial x}I_i(x)\leq\beta<0$ *(or* $\frac{\partial t_i(x)}{\partial x}\equiv 0$*) hold for all* $i=1,2,\dots$ *and* x *from some* ρ_0*-circle of the solution* $y(\tau)$.

Then, if there is an optimal control $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ *for* $\varepsilon\in(0,\varepsilon_0]$ *of the averaged problem* (0.9), (0.11), *then for arbitrary h> 0 there is* $\varepsilon_1=\varepsilon_1(\varepsilon_0,\eta)>0$ *such that*

(1) *for arbitrary* $\varepsilon\in(0,\varepsilon_1)$, *it holds* $|J_\varepsilon^2|<\infty$;

(2) *the inequality* $|J_\varepsilon^2(u_0^*(\varepsilon),v_0^*(\varepsilon))-J_\varepsilon^2|\leq\eta$ *holds.*

Remark 1.3. If under Theorem 1.2 the sets of admissible controls are compact, then optimal control of the averaged problem (0.9), (0.10) exists.

This observation follows from a continuous dependence on the parameters at each finite interval of the solution $y(t,u_0,v_0)$ and Lebesgue, Weierstrass theorems. The proof is based on the corresponding result by A. M. Samoilenko from [6, Theorem 1] for unmanaged impulse systems.

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The Equation in Variations for the Controlled Differential Equation with Delay and its Application

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The controlled differential equations with delay arise in different areas of natural sciences and economics. To illustrate this, below we will consider the simplest model of economic growth. Let $p(t)$ be a quantity of a product produced at the moment t expressed in money units. The fundamental principle of the economic growth has the form

$$p(t) = a(t) + i(t), \tag{1}$$

where $a(t)$ is the so-called apply function and $i(t)$ is a quantity induced investment. We consider the case where the functions $a(t)$ and $i(t)$ have the form

$$a(t) = u_1(t)p(t) \tag{2}$$

and

$$i(t) = u_2(t)p(t-\tau) + \alpha\dot{p}(t), \tag{3}$$

where $u_i(t) \in (0,1)$ for $i = 1,2$, are control functions, $\alpha > 0$ is a given number and $\tau > 0$ is so-called delay parameter.

Formula (3) shows that the value of investment at the moment t depends on the quantity of money at the moment $t-\tau$ (in the past) and on the velocity (production current) at the moment t. From formulas (1)–(3) we get the delay controlled differential equation

$$\dot{p}(t) = \frac{1-u_1(t)}{\alpha}\,p(t) - \frac{u_2(t)}{\alpha}\,p(t-\tau). \tag{4}$$

Let $I = [t_0, t_1]$ be a given interval, suppose that $O \subset \mathbb{R}^n$ is an open set and $U \subset \mathbb{R}^r$ is a compact set. Let the n-dimensional function $f(t,x,y,u,v)$ be continuous on $I \times O^2 \times U^2$ and continuously differentiable with respect to x, y and u, v. Furthermore, let $\tau_2 > \tau_1 > 0$ and $\theta > 0$

be given numbers; let Φ be a set of continuously differentiable functions $\varphi : I_1 = [\widehat{\tau}, t_0] \to O$, where $\widehat{\tau} = t_0 - \tau_2$ and let Ω be a set of piecewise-continuous functions $u(t) \in U$, $t \in I_2 = [\widehat{\theta}, t_1]$, where $\widehat{\theta} = t_0 - \theta$. To each element $\mu = (\tau, \varphi, u) \in \Lambda := [\tau_1, \tau_2] \times \Phi \times \Omega$ we assign the delay controlled differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t), u(t-\theta)\big), \ \ t \in (t_0, t_1) \tag{5}$$

with the initial condition

$$x(t) = \varphi(t), \ \ t \in I_1. \tag{6}$$

Definition. Let $\mu = (\tau, \varphi, u) \in \Lambda$. A function $x(t;\mu) \in O$ for $t \in I_3 = [\widehat{\tau}, t_1]$, is called a solution of equation (5) with the initial condition (6), or a solution corresponding to the element μ and defined on the interval I_3, if $x(t;\mu)$ satisfies condition (6), is absolutely continuous on the interval I and it satisfies equation (5) almost everywhere on (t_0, t_1).

Let us introduce notations

$$|\mu| = |\tau| + \|\varphi\|_1 + \|u\|, \quad \Lambda_\varepsilon(\mu_0) = \big\{\mu \in \Lambda : \ |\mu - \mu_0| \le \varepsilon\big\},$$

where

$$\|\varphi\|_1 = \sup\big\{|\varphi(t)| + |\dot{\varphi}(t)| : \ t \in I_1\big\}, \quad \|u\| = \sup\big\{|u(t)| : \ t \in I_2\big\},$$

$\varepsilon > 0$ is a fixed number and $\mu_0 = (\tau_0, \varphi_0, u_0) \in \Lambda$ is a fixed initial element; furthermore,

$$\delta\tau = \tau - \tau_0, \ \ \delta\varphi(t) = \varphi(t) - \varphi_0(t), \ \ \delta u(t) = u(t) - u_0(t),$$
$$\delta\mu = \mu - \mu_0 = (\delta\tau, \delta\varphi, \delta u), \ \ |\delta\mu| = |\delta\tau| + \|\delta\varphi\|_1 + \|\delta u\|.$$

Theorem. *Let $x_0(t) := x(t;\mu_0)$ be the solution corresponding to the initial element $\mu_0 = (\tau_0, \varphi_0, u_0) \in \Lambda$ and defined on the interval I_3, where $\tau_0 \in (\tau_1, \tau_2)$. Then, there exists $\varepsilon_1 > 0$ such that for each perturbed element $\mu \in \Lambda_{\varepsilon_1}(\mu_0)$ there corresponds the solution $x(t;\mu)$ defined on the interval I_3 and the following representation holds*

$$x(t;\mu) = x_0(t) + \delta x(t;\delta\mu) + o(t;\delta\mu), \ \ t \in (t_0, t_1), \tag{7}$$

where

$$\lim_{|\delta\mu| \to 0} \frac{|o(t;\delta\mu)|}{|\delta\mu|} = 0 \ \ \textit{uniformly for} \ \ t \in (t_0, t_1).$$

Moreover, the function

$$\delta x(t) = \begin{cases} \delta\varphi(t), & t \in I_1, \\ \delta x(t;\delta\mu), & t \in (t_0, t_1) \end{cases}$$

is a solution to the "equation in variations"

$$\dot{\delta x}(t) = f_x[t]\delta x(t) + f_y[t]\delta x(t-\tau_0) - f_y[t]\dot{x}_0(t-\tau_0)\delta\tau + f_u[t]\delta u(t) + f_v[t]\delta u(t-\theta), \ \ t \in (t_0, t_1) \tag{8}$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \ \ t \in [\widehat{\tau}, t_0]. \tag{9}$$

Here $f_x[t] = f_x(t, x_0(t), x_0(t-\tau_0), u_0(t), u_0(t-\theta))$.

The theorem is proved by the scheme given in [1]. Formula (7) and equation (8) allow us to obtain an approximate solution of the perturbed equation (5) in analytical form. In fact, for a small $|\delta\mu|$, from (7) it follows that

$$x(t;\mu) \approx x_0(t) + \delta x(t;\delta\mu), \quad t \in (t_0, t_1). \tag{10}$$

For the economical model (4), where $u_0(t) = (u_{10}(t), u_{20}(t))$ in the initial element $\mu_0 = (\tau_0, \varphi_0, u_0)$ and $p_0(t) = p(t;\mu_0)$, the equation in variations and the initial condition, respectively, have the forms

$$\begin{aligned} \dot{\delta p}(t) = \frac{1-u_{10}(t)}{\alpha}\,\delta p(t) - \frac{u_{20}(t)}{\alpha}\,\delta p(t-\tau_0) \\ + \frac{u_{20}(t)}{\alpha}\,\dot p_0(t-\tau_0)\delta\tau - \frac{p_0(t)}{\alpha}\,\delta u_1(t) - \frac{p_0(t-\tau_0)}{\alpha}\,\delta u_2(t), \quad t \in (t_0, t_1) \end{aligned}$$

and

$$\delta p(t) = \delta\varphi(t), \quad t \in [\widehat{\tau}, t_0].$$

Below, on the basis of formula (10) an approximate solution is constructed for the perturbed equation.

Example.

(a) Let $t_0 = 0$, $t_1 = 2$, $\tau_1 = 0.5$, $\tau_2 = 1.5$, $\tau_0 = 1$, $\varphi_0(t) \equiv 1$,

$$u_0(t) = \begin{cases} \sqrt{2(t+1)^2+1}, & t \in [0,1], \\ \sqrt{2(t+1)^2+t^2}, & t \in [1,2], \end{cases}$$

i.e., in this case $\mu_0 = (1, 1, u_0)$. Consider the scalar original equation

$$\dot x(t) = 2x^2(t) + x^2(t-1) - u_0^2(t) + 1, \quad t \in (0,2),$$

with the initial condition

$$x(t) = 1, \quad t \in [-1.5, 0].$$

It is easy to see that

$$x_0(t) := x(t;\mu_0) = \begin{cases} 1, & t \in [-1.5, 0], \\ t+1, & t \in [0,2]. \end{cases}$$

(b) The perturbed equation

$$\dot x(t) = 2x^2(t) + x^2(t-1-\rho_1) - [u_0(t) + \rho_3\ \sin(t)]^2 + 1, \quad t \in (0,2),$$

with the perturbed initial condition

$$x(t) = 1 + 2\rho_2\cos(t), \quad t \in [-1.5, 0],$$

where $|\rho_i|$ for $i = 1,2,3$ are small fixed numbers. In this case we have

$$\begin{gathered} \mu = (1+\rho_1, 1+2\rho_2\cos(t), u_0(t) + \rho_3\sin(t)), \\ \delta\tau = \rho_1, \delta\varphi(t) = 2\rho_2\cos(t), \quad \delta u(t) = \rho_3\sin(t). \end{gathered}$$

(c) It is clear that

$$f_x[t] = 4x_0(t) = 4(t+1), \quad f_y[t] = 2x_0(t-1), \quad f_u[t] = -2u_0(t).$$

Thus, (8) and (9), respectively, have the forms

$$\dot{\delta x}(t) = 4(t+1)\delta x(t) + 2x_0(t-1)\delta x(t-1) - 2\rho_1 x_0(t-1)\dot{x}_0(t-1) - 2\rho_3 \sin(t) u_0(t)$$

and

$$\delta x(t) = 2\rho_2 \cos(t), \quad t \in [-1.5, 0].$$

By elementary calculations we obtain

$$\delta x(t; \delta\mu) = \begin{cases} \delta x_1(t), & t \in [0,1), \\ \delta x_2(t), & t \in [1,2), \end{cases}$$

where

$$\begin{aligned}
\delta x_1(t) &= 2\Big\{ e^{2t(t+2)} \Big[\rho_2 + \int\limits_0^t e^{-2s(s+2)} \Big(2\rho_2 \cos(s-1) - \rho_3 \sin(s) \sqrt{2(s+1)^2+1} \Big)\, ds \Big] \Big\}, \\
\delta x_2(t) &= e^{2(t^2+2t-3)} \\
&\quad \times \Big\{ \delta x_1(1) + \int\limits_1^t e^{-2(s^2+2s-3)} \Big(2s\delta x_1(s-1) - 2\rho_1 s - 2\rho_3 \sin(s) \sqrt{2(s+1)^2+s^2} \Big)\, ds \Big\}.
\end{aligned}$$

Consequently, the approximate solution $x(t;\mu)$ of the perturbed equation has the form (see (10))

$$x(t;\mu) \approx t + 1 + \delta x(t; \delta\mu), \quad t \in (0,2).$$

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Asymptotic Analysis of Two-Dimensional Cyclic Systems of First Order Nonlinear Differential Equations

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1 Introduction

This paper is concerned with positive solutions of the two-dimensional cyclic systems of first order nonlinear differential equations of the forms

(A) $x' + p(t)y^{\alpha} = 0$, $y' - q(t)x^{\beta} = 0$, $t \geqq a$;

(B) $x' - p(t)y^{\alpha} = 0$, $y' + q(t)x^{\beta} = 0$, $t \geqq a$

for which the following conditions are always assumed to hold:

(a) α and β are positive constants such that $\alpha\beta < 1$;

(b) $p, q : [a, \infty) \to (0, \infty)$, $a \geqq 0$ are regularly varying functions such that

$$p(t) = t^{\lambda} l(t), \;\; q(t) = t^{\mu} m(t), \;\; l, m \in \mathrm{SV}.$$

By a positive solution of (A) or (B) we mean a vector function $(x(t), y(t))$ both components of which are positive and satisfy the system (A) or (B) in a neighborhood of infinity. In this paper we are concerned with exclusively with positive solutions of (A) and (B) both components of which are regularly varying functions in the sense of Karamata. Such a solution $(x(t), y(t))$ is called regularly varying of index (ρ, σ) if $x(t)$ and $y(t)$ are regularly varying of indices $\rho(\in \mathbb{R})$ and $\sigma(\in \mathbb{R})$, respectively, and is denoted by $(x, y) \in \mathrm{RV}(\rho, \sigma)$.

Since the publication of the book [3] of Marić in the year 2000, the class of regularly varying functions in the sense of Karamata is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x'' = q(t)x, \;\; q(t) > 0.$$

The definitions and properties of regularly varying functions

Definition 1.1. A measurable function $f : [a, \infty) \to (0, \infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \;\text{ for any }\; \lambda > 0, \;\; \rho \in \mathbb{R}.$$

Propsoition 1.1 (Representation Theorem)**.** *A measurable function $f : [a, \infty) \to (0, \infty)$ is regularly varying of index ρ if and only if it can be written in the form*

$$f(t) = c(t) \exp\bigg\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \bigg\}, \;\; t \geqq t_0,$$

for some $t_0 > a$, *where* $c(t)$ *and* $\delta(t)$ *are measurable functions such that*

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \textit{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho L(t)$ for some $L \in \mathrm{SV}$. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ \ (m_i \in \mathbb{R}), \quad \exp\Big\{\prod_{i=1}^{N} (\log_i t)^{n_i}\Big\} \ \ (0 < n_i < 1), \quad \exp\Big\{\frac{\log t}{\log_2 t}\Big\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \dots, N$, also belong to the set of slowly varying functions.

Propsoition 1.2. *Let* $L(t)$ *be any slowly varying function. Then, for any* $\gamma > 0$,

$$\lim_{t\to\infty} t^\gamma L(t) = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels [1].

2 Main results

The papers [2] and [4] are devoted to the analysis of strongly decreasing and increasing regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ of the system

(C) $\quad x' + p(t) y^\alpha = 0, \ y' + q(t) x^\beta = 0, \ t \geqq a;$

(D) $\quad x' - p(t) y^\alpha = 0, \ y' - q(t) x^\beta = 0, \ t \geqq a.$

(More precisely, $\rho < 0$ and $\sigma < 0$, $\rho = 0$ and $\sigma < 0$, $\rho < 0$ and $\sigma = 0$ for system (C), moreover, $\rho > 0$ and $\sigma > 0$, $\rho = 0$ and $\sigma > 0$, $\rho > 0$ and $\sigma = 0$ for system (D).) The purpose of this talk is to supplement necessary and sufficient conditions for the existence of regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ of (A) and (B) with either $\rho = 0$ or $\sigma = 0$, in which case either $x(t)$ or $y(t)$ is slowly varying, and then to determine their asymptotic behavior as $t \to \infty$ accurately. Our main results are the following.

Theorem 2.1. *System* (A) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho = 0$ *and* $\sigma > 0$ *such that* $\lim\limits_{t\to\infty} x(t) = 0$ *and* $\lim\limits_{t\to\infty} y(t) = \infty$ *if and only if*

$$\lambda + 1 + \alpha(\mu + 1) = 0, \ \ \mu + 1 > 0$$

and

$$\int_a^\infty p(t) (t q(t))^\alpha \, dt < \infty,$$

in which case $\sigma = \mu + 1$ *and any such solution* $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim \bigg[(1-\alpha\beta)\int\limits_t^\infty p(s)\Big(\frac{sq(s)}{\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty,$$

$$y(t) \sim \frac{tq(t)}{\sigma}\,\bigg[(1-\alpha\beta)\int\limits_t^\infty p(s)\Big(\frac{sq(s)}{\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}}, \ \ t \to \infty,$$

where the symbol $\sim$ *is used to denote the asymptotic equivalence*

$$f(t) \sim g(t) \ \ \text{as} \ \ t \to \infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Theorem 2.2. *System* (A) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho < 0$ *and* $\sigma = 0$ *such that* $\lim\limits_{t\to\infty} x(t) = 0$ *and* $\lim\limits_{t\to\infty} y(t) = \infty$ *if and only if*

$$\lambda + 1 < 0, \ \ \beta(\lambda + 1) + \mu + 1 = 0$$

and

$$\int\limits_a^\infty (tp(t))^\beta q(t)\,dt = \infty,$$

in which case $\rho = \lambda + 1$ *any such solution* $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim -\frac{tp(t)}{\rho}\,\bigg[(1-\alpha\beta)\int\limits_a^t \Big(\frac{sp(s)}{-\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{\alpha}{1-\alpha\beta}}, \ \ t \to \infty,$$

$$y(t) \sim \bigg[(1-\alpha\beta)\int\limits_a^t \Big(\frac{sp(s)}{-\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty.$$

Theorem 2.3. *System* (A) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho < 0$ *and* $\sigma > 0$ *such that* $\lim\limits_{t\to\infty} x(t) = 0$ *and* $\lim\limits_{t\to\infty} y(t) = \infty$ *if and only if*

$$\lambda + 1 + \alpha(\mu + 1) < 0, \ \ \beta(\lambda + 1) + \mu + 1 > 0,$$

in which case

$$\rho = \frac{\lambda + 1 + \alpha(\mu+1)}{1-\alpha\beta}\,, \ \ \sigma = \frac{\beta(\lambda+1) + \mu + 1}{1-\alpha\beta}$$

and any such solution $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^\alpha}{-\rho\sigma^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^\beta q(t)}{(-\rho)^\beta\sigma}\Big]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty.$$

Theorem 2.4. *System* (B) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho = 0$ *and* $\sigma < 0$ *such that* $\lim\limits_{t\to\infty} x(t) = \infty$ *and* $\lim\limits_{t\to\infty} y(t) = 0$ *if and only if*

$$\lambda + 1 + \alpha(\mu + 1) = 0, \ \ \mu + 1 < 0$$

and

$$\int\limits_a^\infty p(t)(tq(t))^\alpha\,dt = \infty,$$

in which case $\sigma = \mu + 1$ *and any such solution* $(x(t), y(t))$ *of* (B) *has one and the same asymptotic behavior*

$$x(t) \sim \bigg[(1-\alpha\beta)\int\limits_a^t p(s)\Big(\frac{sq(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty,$$

$$y(t) \sim -\frac{tq(t)}{\sigma}\,\bigg[(1-\alpha\beta)\int\limits_a^t p(s)\Big(\frac{sq(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}}, \ \ t \to \infty.$$

Theorem 2.5. *System* (B) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho > 0$ *and* $\sigma = 0$ *such that* $\lim\limits_{t\to\infty} x(t) = \infty$ *and* $\lim\limits_{t\to\infty} y(t) = 0$ *if and only if*

$$\lambda + 1 > 0, \ \ \beta(\lambda+1) + \mu + 1 = 0$$

and

$$\int\limits_a^\infty (tp(t))^\beta q(t)\,dt < \infty,$$

in which case $\rho = \lambda + 1$ *and any such solution* $(x(t), y(t))$ *of* (B) *has one and the same asymptotic behavior*

$$x(t) \sim \frac{tp(t)}{\rho}\,\bigg[(1-\alpha\beta)\int\limits_t^\infty \Big(\frac{sp(s)}{\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{\alpha}{1-\alpha\beta}}, \ \ t \to \infty,$$

$$y(t) \sim \bigg[(1-\alpha\beta)\int\limits_t^\infty \Big(\frac{sp(s)}{\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty.$$

Theorem 2.6. *System* (B) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho > 0$ *and* $\sigma < 0$ *such that* $\lim\limits_{t\to\infty} x(t) = \infty$ *and* $\lim\limits_{t\to\infty} y(t) = 0$ *if and only if*

$$\lambda + 1 + \alpha(\mu+1) > 0, \ \ \beta(\lambda+1) + \mu + 1 < 0,$$

in which case

$$\rho = \frac{\lambda + 1 + \alpha(\mu+1)}{1-\alpha\beta}, \quad \sigma = \frac{\beta(\lambda+1) + \mu + 1}{1-\alpha\beta}$$

and any such solution $(x(t), y(t))$ *of* (B) *has one and the same asymptotic behavior*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^\alpha}{\rho(-\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}}, \quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^\beta q(t)}{-\rho^\beta\sigma}\Big]^{\frac{1}{1-\alpha\beta}}, \ \ t \to \infty.$$

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Set of Points of Lower Semicontinuity for the Topological Entropy of a Family of Dynamical Systems Continuously Depending on a Parameter

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Let us give a precise definition of topological entropy [1]. Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous mapping. Along with the original metric d, we define an additional system of metrics

$$d_n^f(x,y) = \max_{0 \le i \le n-1} d(f^i(x), f^i(y)), \ \ x, y \in X, \ \ n \in \mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of the mapping f, $f^0 \equiv \mathrm{id}_X$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, by $N_d(f, \varepsilon, n)$ we denote the maximum number of points in X such that the pairwise d_n^f-distances between them are greater than ε. Such a set of points is said to be (f, ε, n)-separated. Then the *topological entropy* of the dynamical system generated by the continuous mapping f is defined as the quantity (which may be a nonnegative real number or infinity)

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln N_d(f, \varepsilon, n). \tag{1}$$

Note that the quantity (1) remains unchanged if the metric d in its definition is replaced by any other metric that defines the same topology on X as d; this, in particular, explains why the entropy (1) is said to be topological.

Given a metric space $\mathcal{M}$ and a jointly continuous mapping

$$f : \mathcal{M} \times X \to X \tag{2}$$

we form the function

$$\mu \longmapsto h_{\mathrm{top}}(f(\mu, \cdot\,)). \tag{3}$$

Recall that a point μ_0 of the metric space $\mathcal{M}$ is called a point of lower semicontinuity of a function $h : \mathcal{M} \to \mathbb{R} \cup \{\infty\}$ if for each sequence $(\mu_k)_{k \in \mathbb{N}}$ of points in $\mathcal{M}$ converging to μ_0, one has the inequality

$$h(\mu_0) \le \underline{\lim_{k \to +\infty}} h(\mu_k).$$

It was proved in [3] that if the space $\mathcal{M}$ is complete, then the property of lower semicontinuity is Baire typical for the topological entropy of a family of mappings (2); in other words, the set of points of $\mathcal{M}$ at which the function (3) is lower semicontinuous contains a dense G_δ-set in $\mathcal{M}$. It was established in [4] that the set of points of lower semicontinuity is itself an everywhere dense G_δ-set in $\mathcal{M}$. In addition, an example of a mapping (2) (where the parameter space $\mathcal{M}$ is the Cantor perfect set in the interval $[0, 1]$) for which the set of points of lower semicontinuity is not an F_σ-set was constructed in [4].

By definition [2, p. 277], a metric space has dimension zero if any of its points has an arbitrarily small neighborhood that is simultaneously closed and open, which is equivalent to the emptiness of the boundary of this neighborhood. One example of such a space is the Cantor perfect set $\mathcal{K}$ (the set of infinite ternary fractions $x = 0,\, a_1a_2a_3, \dots$, where $a_i \in \{0,\, 2\}$) in the interval $[0, 1]$ with the metric induced by the natural metric of the real line.

A natural question arises: what is the set of lower semicontinuity points of a function (3). In the paper [5] we derived a complete description of the set of points of lower semicontinuity of a function (3) for each complete metric separable zero-dimensional space $\mathcal{M}$.

For an open everywhere dense subset of a complete metric separable zero-dimensional space $\mathcal{M}$ the following theorem holds.

Theorem 1. *Let $\mathcal{M}$ be a complete separable zero-dimensional metric space and let $X = \mathcal{K}$ be the Cantor perfect set in the interval $[0, 1]$ with the metric induced by the natural metric of the real line. Then for each open everywhere dense subset G of the space $\mathcal{M}$ there exists a mapping* (2) *such that the function* (3) *is bounded and its set of points of lower semicontinuity coincides with the set G.*

For an open everywhere dense G_δ-subset of a complete metric separable zero-dimensional space $\mathcal{M}$ the following theorem holds.

Theorem 2. *Let $\mathcal{M}$ be a complete separable zero-dimensional metric space and let $X = \mathcal{K}$ be the Cantor perfect set in the interval $[0, 1]$ with the metric induced by the natural metric of the real line. Then for each everywhere dense G_δ-subset G of the space $\mathcal{M}$ there exists a mapping* (2) *such that the set of points of lower semicontinuity of the function* (3) *coincides with the set G.*

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