On the Sets of Lower Semicontinuity Points and Upper Semicontinuity Points of Topological Entropy with Continuous Dependence on a Parameter

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1 Statement of the problems

Let us present definitions needed in what follows. Let X be a compact metric space with the metric d. Take a continuous mapping $f : X \to X$. By $f^{\circ n}$ we denote the n-th iteration of f, i.e.,

$$f^{\circ n} = \underbrace{f \circ \cdots \circ f}_{n}, \ \ n = 0, 1, 2, \dots;$$

$f^{\circ 0} \equiv \mathrm{id}$ by the definition. Along with the original metric d, we introduce a nondecreasing sequence $(d_n^f)_{n\in\mathbb{N}}$ of metrics on X defined by the equality

$$d_n^f(x, y) = \max_{0\le i\le n-1} d\big(f^{\circ i}(x), f^{\circ i}(y)\big), \ \ n \in \mathbb{N}, \ \ x, y \in X.$$

By $B_f(x, \varepsilon, n)$ we denote the open ball with the center x and radius ε in the metric d_n^f, i.e.,

$$B_f(x, \varepsilon, n) = \big\{y \in X : \ d_n^f(x, y) < \varepsilon\big\}.$$

A set $E \subset X$ is called an (f, ε, n)-cover if

$$X \subset \bigcup_{x\in E} B_f(x, \varepsilon, n).$$

For each (f, ε, n)-cover we find the number of its elements; let $S_d(f, \varepsilon, n)$ be the least of these numbers. The *topological entropy* of the dynamical system generated by a continuous mapping f is defined as follows [1]:

$$h_{\mathrm{top}}(f) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \ln S_d(f, \varepsilon, n). \tag{1.1}$$

Note that the topological entropy is independent of the choice of a metric generating the given topology on X and hence is well defined by (1.1).

Given a metric space $\mathcal{M}$ and a jointly continuous map

$$f : \mathcal{M} \times X \to X \tag{1.2}$$

we define the function

$$\mu \longmapsto h_{\mathrm{top}}(f_\mu(\,\cdot\,)). \tag{1.3}$$

It was proved in [3] that, in the case of $X=[0;1]$, the function (1.3) is lower semicontinuous. In the general case (for arbitrary X), the function (1.3) is not necessarily lower semicontinuous. For example, consider the family of maps $f_\mu : X_1 \to X_1$, where

$$X_1=\big\{z\in \mathbf{C}:\ |z|\leqslant 1\big\},\quad f_\mu(z)=\begin{cases} 0 & \text{if } z=0,\\ \mu\,\dfrac{z^2}{|z|} & \text{if } z\neq 0,\end{cases}\qquad \mu\in[0;1].$$

Take a $\mu\in[0;1)$ and an $\varepsilon>0$. There exists a positive integer $n(\mu,\varepsilon)$ such that

$$d\big(f_\mu^i(z),f_\mu^i(w)\big)\leqslant d\big(f_\mu^i(z),0)+d(0,f_\mu^i(w)\big)\leqslant 2\mu^i<\varepsilon,$$

for any $i\geqslant n(\mu,\varepsilon)$ and any points $z,w\in X$; therefore, for any positive integer $n\geqslant n(\mu,\varepsilon)$ we have

$$d_n^{f_\mu}(z,w)=\max_{0\leqslant i\leqslant n-1} d\big(f_\mu^i(z),f_\mu^i(w)\big)\leqslant \max\big\{d_{n(\mu,\varepsilon)}^{f_\mu}(z,w),\varepsilon\big\}.$$

Hence if $n\geqslant n(\mu,\varepsilon)$, then

$$S_d(f_\mu,\varepsilon,n)\leqslant S_d(f_\mu,\varepsilon,n(\mu,\varepsilon)).$$

It follows that

$$0\leqslant h_{\rm top}(f_\mu)=\lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\frac{1}{n}\,\ln S_d(f_\mu,\varepsilon,n)\leqslant \lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\frac{1}{n}\,\ln S_d(f_\mu,\varepsilon,n(\mu,\varepsilon))=0.$$

Thus, for $\mu\in[0;1)$ we have $h_{\rm top}(f_\mu)=0$.

For each positive integer $k\geqslant 4$, we set

$$\varepsilon_k=\sqrt{2\Big(1-\cos\Big(\frac{2\pi}{2^k}\Big)\Big)}\,.$$

Given a positive integer $n\geqslant 4$, consider the set

$$\mathcal{Z}=\Big\{z_m=\exp\Big(\frac{2\pi mi}{2^{k+n}}\Big)\Big\},\ \ m=0,\dots,2^{k+n}-1.$$

If the distance between two points z_p and z_q of $\mathcal{Z}$ satisfies the inequality $d(z_p,z_q)\geqslant\varepsilon_k$, then $d_n^{f_1}(z_p,z_q)\geqslant\varepsilon_k$, and if the distance between z_p and z_q satisfies the inequality $d(z_p,z_q)<\varepsilon_k$, then there exists an $l\leqslant n-1$ such that

$$d_n^{f_1}(z_p,z_q)\geqslant d\big(f_1^l(z_p),f_1^l(z_q)\big)\geqslant\varepsilon_k.$$

Thus, for any two points of $\mathcal{Z}$ we have $d_n^{f_1}(z_p,z_q)\geqslant\varepsilon_k$. This implies

$$S_d(f_1,\varphi_0,\varepsilon_k,n)\geqslant 2^{k+n},$$

whence

$$h_{\rm top}(f_1)=\lim_{k\to\infty}\varlimsup_{n\to\infty}\frac{1}{n}\,\ln S_d(f_1,\varepsilon_k,n)\geqslant \ln 2.$$

Thus, the function $\mu\mapsto h_{\rm top}(f_\mu)$ is discontinuous at $\mu=1$. Moreover, it is not lower semicontinuous at $\mu=1$.

In the present paper we study the sets of upper semicontinuity and lower semicontinuity points of the function (1.3).

2 The typicality of the lower semicontinuity of topological entropy

Theorem 2.1. *If $\mathcal{M}$ is a complete metric space, then for any map* (1.2)*, the set of lower semicontinuity points of the function* (1.3) *is everywhere dense G_δ-set in the space $\mathcal{M}$.*

Consider the Baire space $\mathfrak{B}$

$$\mathfrak{B} = \big\{x = (x_1, x_2, \dots):\ x_k \in \{0, 1\},\ k \in \mathbb{N}\big\}$$

of 0-1-sequences with the metric defined by the formula

$$d(x, y) = \begin{cases} 0 & \text{if } \ x = y, \\ 2^{-\min\{k:\, x_k \neq y_k\}} & \text{if } \ x \neq y. \end{cases}$$

Then the metric space $\mathfrak{B}$ is compact.

Theorem 2.2. *Let $\mathcal{M} = X = \mathfrak{B}$, then for the map*

$$f\big((\mu_1, \mu_2, \dots), (x_1, x_2, \dots)\big) = (x_{1+\mu_1}, x_{2+\mu_2}, \dots)$$

the set of lower semicontinuity points of the function (1.3) *is not an F_σ-set in the space $\mathcal{M}$.*

Let $C(\mathfrak{B}, \mathfrak{B})$ be the space of continuous mappings of $\mathfrak{B}$ into $\mathfrak{B}$ with the metric

$$\varrho(f, g) = \max_{x \in \mathfrak{B}} d(f(x), g(x)).$$

Block [2] found that topological entropy is discontinuous at every point in space $C(\mathfrak{B}, \mathfrak{B})$.

Theorem 2.3. *The set of zeros of the function*

$$h_{\text{top}} : C(\mathfrak{B}, \mathfrak{B}) \to [0, +\infty) \tag{2.1}$$

coincides with the set of its lower semicontinuity points.

From Theorem 2.1 it follows that the set of zeros of the function (2.1) is an everywhere dense G_δ-set in the space $C(\mathfrak{B}, \mathfrak{B})$.

3 Emptiness of the set of upper semicontinuity points of topological entropy

Yomdin [5] and Newhouse [4] proved that the topological entropy of C^∞-diffeomorphisms on a compact Riemannian manifold is upper semicontinuous.

Theorem 3.1. *For any map* (1.2)*, the set of upper semicontinuity points of the function* (1.3) *is an $F_{\sigma\delta}$-set in the space $\mathcal{M}$.*

Theorem 3.2. *Let $\mathcal{M} = X = \mathfrak{B}$, then there exists a map* (1.2) *such that the set of upper semicontinuity points of the function* (1.3) *is empty.*

References

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