On Relations Between Perron and Lyapunov Regularity Coefficients of Parametric Linear Differential Equations

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For any $n \in \mathbb{N}$ we consider the linear system of differential equations

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with a continuous coefficient $n \times n$ matrix uniformly bounded on the time half-line. Along with system (1), consider the adjoint system

$$\dot{y} = -A^{\mathrm{T}}(t)y, \quad y \in \mathbb{R}^n, \quad t \geq 0. \tag{2}$$

Obviously, the adjoint to system (2) is system (1); therefore, systems (1) and (2) are said to be mutually adjoint. Everywhere below, we identify system (1) with its coefficient matrix.

The so-called Perron and Lyapunov regularity coefficients $\sigma_P(A)$ and $\sigma_L(A)$, respectively, defined for each system (1) play an important role in the asymptotic theory of linear differential systems [3,4]. They essentially specify the response of system (1) to linear exponentially decreasing perturbations and nonlinear perturbations of a higher smallness order; in particular, the vanishing of at least one (and hence both) of them is equivalent to the Lyapunov regularity of system (1).

Let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ be the Lyapunov exponents of system (1) arranged in nondescending order, and let $\mu_1(A) \geq \cdots \geq \mu_n(A)$ be the Lyapunov exponents of the adjoint system (2) arranged in nonascending order. By Sp we denote the trace of a matrix. Then, by definition,

$$\sigma_P(A) = \max_{1 \leq i \leq n} \big\{ \lambda_i(A) + \mu_i(A) \big\} \ \text{ and } \ \sigma_L(A) = \sum_{i=1}^{n} \lambda_i(A) - \varliminf_{t \to +\infty} \frac{1}{t} \int_0^t \operatorname{Sp} A(\tau)\, d\tau.$$

It was shown in the monograph [2, § 1] that the regularity coefficients of any n-dimensional system (1) satisfy the inequalities

$$0 \leq \sigma_P(A) \leq \sigma_L(A) \leq n\sigma_P(A). \tag{3}$$

In the paper [5], it has been shown that inequalities (3) describe all possible relations between the regularity coefficients of differential systems. In other words, it was shown that for any positive integer n and ordered pair of numbers $(p; \ell)$ satisfying the inequalities $0 \leq p \leq \ell \leq np$, there exists a system A such that $\sigma_P(A) = p$ and $\sigma_L(A) = \ell$.

Let M be a metric space. Along with the individual system (1) we consider a family of linear differential systems

$$\dot{x} = A(t, \xi)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{4}$$

such that for every $\xi \in M$ the matrix-valued function $A(\,\cdot\,, \xi) \colon [0, +\infty) \to \mathbb{R}^{n \times n}$ is continuous and uniformly bounded on the time half-line, i.e. there exists $a_\xi \in \mathbb{R}$ such that $\sup\limits_{t \in [0, +\infty)} \|A(t, \xi)\| \leq a_\xi$. Moreover, we suppose that the family of matrix-valued functions $A(\,\cdot\,, \xi)$, $\xi \in M$, is continuous in compact-open topology, in other words, if a sequence $(\xi_k)_{k\in\mathbb{N}}$, $\xi_k \in M$, converges to ξ_0, then the sequence of functions $A(\,\cdot\,,\xi_k)$ converges to $A(\,\cdot\,,\xi_0)$ uniformly on every interval of $[0,+\infty)$. For a symbol $\varkappa \in \{P,L\}$ by $\sigma^A_\varkappa(\,\cdot\,)\colon M \to \mathbb{R}$ we denote a function acting by the rule $\xi \mapsto \sigma_\varkappa(A(\,\cdot\,,\xi))$. In a natural way a problem of complete description of pair $(\sigma^A_P(\,\cdot\,),\sigma^A_L(\,\cdot\,))$ arises. First we need introduce some notation to formulate a solution of this problem.

Let $f(\,\cdot\,)$ be a real-valued function defined on some set M. For a number $r \in \mathbb{R}$ and for the function $f(\,\cdot\,)$ the Lebesgue set $[f \geq r]$ is defined as the set $[f \geq r] \stackrel{\text{def}}{=} \{t \in M\colon\; f(t) \geq r\}$. If M is a topological space then G_δ stands for a system of subsets in M which can be represented as countable intersections of open sets. We say [1, pp. 223–224] that a function $f(\,\cdot\,)\colon M \to \mathbb{R}$ belongs to the class $({}^*, G_\delta)$, or $f(\,\cdot\,)$ is a function of the class $({}^*, G_\delta)$ if its Lebesgue set satisfies the condition $[f \geq r] \in G_\delta$ for any $r \in \mathbb{R}$.

Theorem. *For functions $p(\,\cdot\,), \ell(\,\cdot\,)\colon M \to \mathbb{R}$ there exists a parametric system* (4) *such that $\sigma^A_P(\,\cdot\,) \equiv p(\,\cdot\,)$ and $\sigma^A_L(\,\cdot\,) \equiv \ell(\,\cdot\,)$ if and only if $p(\,\cdot\,)$, $\ell(\,\cdot\,)$ are functions of the class $({}^*, G_\delta)$ and for every $\xi \in M$ the following inequalities*

$$0 \leq p(\xi) \leq \ell(\xi) \leq np(\xi)$$

hold.

References

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