On Topological Classifications of Some Classes of Complex Differential Systems

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1 Covering foliations

The foliations theory began with works of H. Poincaré. It have began an independent scientific field and actually is consider as an efficient tool in the topological investigations. Here we consider foliations of a special type, referred to as covering foliations [5]. We will consider the problem of topological classification of covering foliations determinated by the complex linear differential systems and homogeneous projective matrix Riccati equations.

Definition 1.1. Let A and B be path connected smooth varieties of dimensions $\dim A = n$ and $\dim B = m$. Smooth foliation $\mathfrak{F}$ of dimension m on the variety $A\times B$, locally transversal to $A\times\{b\}$ for all $b\in B$, we will name ***a covering foliation***, if the projection $p\colon A\times B\to B$ on the second factor defines for each layer of it foliation covering of the variety B.

Definition 1.2. Let $\mathfrak{F}_c$ be a layer of the covering foliation $\mathfrak{F}$, containing the point $c\in A\times B$. ***The phase group*** $Ph(\mathfrak{F},b_0)$, $b_0\in B$, of the covering foliation $\mathfrak{F}$ we will name the group of the diffeomorphisms $\operatorname{Diff}(A,\pi_1(B,b_0))$ of the actions on the phase layer A by fundamental group $\pi_1(B,b_0)$ with noted point b_0, defined under formulae $\Phi^\gamma(a) = q\circ r\circ s$ for all $a\in A$, for all $\gamma\in\pi_1(B,b_0)$, where r is a lifting of one of ways $s(\tau)\subset B$ for all $\tau\in[0,1]$, corresponding to the element γ of the group $\pi_1(B,b_0)$, on the layer $\mathfrak{F}_{(a,s(0))}$ of the covering foliation $\mathfrak{F}$ in the point $(a,s(0))$, and $q: A\times B\to A$ is a projection to the first factor.

It is easy to see that owing to path connectivity and smoothness of the variety B, then phase groups $Ph(\mathfrak{F},b_1)$ and $Ph(\mathfrak{F},b_2)$ are smoothly conjugated for any two points b_1 and b_2 of the base B. Therefore further we will speak simply about of ***the phase group*** $Ph(\mathfrak{F})$ of the covering foliation $\mathfrak{F}$, not connecting it with any point of the base B.

Definition 1.3. We will say that the covering foliation $\mathfrak{F}^1$ on the variety $A_1\times B_1$ is ***topologically equivalent*** to the covering foliation $\mathfrak{F}^2$ on the variety $A_2\times B_2$ if there exists the homeomorphism $h: A_1\times B_1\to A_2\times B_2$ such that $q_2\circ h(A_1\times B_1) = A_2$, $h(\mathfrak{F}^1_{c_1}) = \mathfrak{F}^2_{h(c_1)}$ for all $c_1\in A_1\times B_1$, where q_2 is a projection to the first factor.

Definition 1.4. Let $\mathfrak{F}(\lambda)$ is a smooth family of covering foliations, $\mathfrak{F}(\lambda^0)=\mathfrak{F}$, $\lambda=(\lambda_1,\dots,\lambda_l)$. We will say that the covering foliations $\mathfrak{F}$ is ***structurally stable*** if for all enough small δ any covering foliation $\mathfrak{F}(\lambda)$ is topologically equivalent to it, where norm $\|\lambda-\lambda^0\|<\delta$.

Theorem 1.5. *For topological equivalence of the covering foliations $\mathfrak{F}^1$ and $\mathfrak{F}^2$ it is necessary and sufficient existence of the isomorphism μ of the fundamental groups $\pi_1(B_1)$ and $\pi_1(B_2)$, generated by the homeomorphism $g_\mu: B_1\to B_2$ of the bases, and existence of the homeomorphism $f: A_1\to A_2$ of phase layers such that $f\circ\Phi_1^{\gamma_1} = \Phi_2^{\mu(\gamma_1)}\circ f$ for all $\gamma_1\in\pi_1(B_1)$, where $\Phi_\xi^{\gamma_\xi}\in Ph(\mathfrak{F}^\xi)$, $\gamma_\xi\in\pi_1(B_\xi)$, $\xi=1,2$.*

Concepts of smooth and real holomorphic equivalence of covering foliations are similarly introduced. Also corresponding analogues of Theorem 1.5 are similarly proved.

2 Complex nonautonomous linear differential systems

Consider the complex nonautonomous linear differential systems

$$dw = \sum_{j=1}^{m} A_j(z_1, \dots, z_m) w \, dz_j \tag{2.1}$$

and

$$dw = \sum_{j=1}^{m} B_j(z_1, \dots, z_m) w \, dz_j, \tag{2.2}$$

ordinary at $m = 1$ and completely solvable at $m > 1$, where $w = (w_1, \dots, w_n)$, square matrices $A_j(z_1, \dots, z_m) = \|a_{ikj}(z_1, \dots, z_m)\|$ and $B_j(z_1, \dots, z_m) = \|b_{ikj}(z_1, \dots, z_m)\|$ of the order n consist from holomorphic functions $a_{ikj} : A \to \mathbb{C}$ and $b_{ikj} : B \to \mathbb{C}$, $i = 1, \dots, n$, $k = 1, \dots, n$, $j = 1, \dots, m$, path connected holomorphic varieties A and B are holomorphically equivalent each other. The general solutions of systems (2.1) and (2.2) define covering foliations L^1 and L^2, accordingly, on the varieties $\mathbb{C}^n \times A$ and $\mathbb{C}^n \times B$. The phase group $Ph(L^1)$ of the covering foliation L^1 is generated by the forming nondegenerate linear transformations $P_r w$ for all $w \in \mathbb{C}^n$, $P_r \in GL(n, \mathbb{C})$, $r \in I$, and the phase group $Ph(L^2)$ of the covering foliation L^2 is generated by the forming nondegenerate linear transformations $Q_r w$ for all $w \in \mathbb{C}^n$, $Q_r \in GL(n, \mathbb{C})$, for all $r \in I$, where I is some set of indexes. Also the phase group $Ph(L^1)$ (the phase group $Ph(L^2)$) define the monodromy group of system (2.1) (system (2.2)). In the case $n = 1$, topological equivalence of the scalar equations (2.1) and (2.2) is studied in article [3]. Notice that it is a case integrated in quadratures. We will assume further $n > 1$.

Definition 2.1. A set $\{\lambda_1, \dots, \lambda_n\}$ of nonzero complex numbers we will name ***simple*** if $\lambda_k \setminus \lambda_l \notin s_{lk}^{\pm 1}$, $s_{lk} \in \mathbb{N}$, $l \neq k$, $k = 1, \dots, n$, $l = 1, \dots, n$, and a square matrix of the size $n > 1$ we will name ***simple*** if it has simple structure and simple collection of eigenvalues.

Theorem 2.2. *Let the matrices* $P_r = S \operatorname{diag}\{p_{1r}, \dots, p_{nr}\} S^{-1}$, $Q_r = T \operatorname{diag}\{q_{1r}, \dots, q_{nr}\} T^{-1}$, *and the matrixes* $\ln P_r$ *and* $\ln Q_r$ *be simple for all* $r \in I$. *Then for the topological equivalence of systems* (2.1) *and* (2.2) *it is necessary and sufficient existence of such permutations* $\mu : I \to I$, $\varrho : (1, \dots, n) \to (1, \dots, n)$ *and complex numbers* α_k *with* $\operatorname{Re} \alpha_k > -1$, $k = 1, \dots, n$, *that either* $q_{\varrho(k)\mu(r)} = p_{kr} |p_{kr}|^{\alpha_k}$ *for all* $r \in I$, *or* $q_{\varrho(k)\mu(r)} = \overline{p}_{kr} |p_{kr}|^{\alpha_k}$ *for all* $r \in I$, $k = 1, \dots, n$.

Theorem 2.3. *From a topological equivalence of systems* (2.1) *and* (2.2) *with the non-Abelian monodromy groups of general situation follows their real holomorphic equivalence.*

Theorem 2.4. *Systems* (2.1) *and* (2.2) *are smooth* (*real holomorphic*) *equivalent if and only if its monodromy groups are* $\mathbb{R}$*-linearly conjugated for some permutation* $\mu : I \to I$.

Theorem 2.5. *System* (2.1) *is structurally stable if and only if it monodromy group have one independent generator and the conditions of Theorem* 2.2 *are fulfilled for the matrix* P_1.

3 Complex nonautonomous homogeneous projective matrix Riccati equations

Consider the complex nonautonomous homogeneous projective matrix Riccati equations [5]

$$dv = \sum_{j=1}^{m} A_j(z_1, \dots, z_m) v \, dz_j \tag{3.1}$$

and
$$dv = \sum_{j=1}^{m} B_j(z_1,\dots,z_m)v\,dz_j. \tag{3.2}$$
ordinary at $m=1$ and completely solvable at $m>1$, where $v=(v_1,\dots,v_{n+1})$ are homogeneous coordinates, square matrices $A_j(z_1,\dots,z_m)=\|a_{ikj}(z_1,\dots,z_m)\|$ and $B_j(z_1,\dots,z_m)=\|b_{ikj}(z_1,\dots,z_m)\|$ of the order $n+1$ consist from holomorphic functions $a_{ikj}:A\to\mathbb{C}$ and $b_{ikj}:B\to\mathbb{C}$, $i=1,\dots,n+1$, $k=1,\dots,n+1$, $j=1,\dots,m$, path connected holomorphic varieties A and B are holomorphically equivalent each other. The general solutions of systems (3.1) and (3.2) define covering foliations PL^1 and PL^2, accordingly, on the varieties $\mathbb{C}P^n\times A$ and $\mathbb{C}P^n\times B$. The phase group $Ph(PL^1)$ of the covering foliation PL^1 is generated by the forming nondegenerate linear-fractional transformations P_rv for all $v\in\mathbb{C}P^n$, $P_r\in GL(n+1,\mathbb{C})$, $r\in I$, and the phase group $Ph(PL^2)$ of the covering foliation PL^2 is generated by the forming nondegenerate linear-fractional transformations Q_rv for all $v\in\mathbb{C}P^n$, $Q_r\in GL(n+1,\mathbb{C})$, for all $r\in I$, where I is some set of indexes. Also the phase group $Ph(L^1)$ (the phase group $Ph(L^2)$) define the holonomy group of system (3.1) (system (3.2)).

Theorem 3.1. *Let at $n=1$ the matrices $P_r=S\operatorname{diag}\{p_{1r},p_{2r}\}\,S^{-1}$ for all $r\in I$, $Q_r=T\operatorname{diag}\{q_{1r},q_{2r}\}\,T^{-1}$ for all $r\in I$. Then for the topological equivalence of systems* (3.1) *and* (3.2) *it is necessary and sufficient existence of such permutation $\mu:I\to I$ and complex number α with* $\operatorname{Re}\alpha\neq-1$ *that either*
$$\frac{q_{1r}}{q_{2r}}=\frac{p_{1r}}{p_{2r}}\Big|\frac{p_{1r}}{p_{2r}}\Big|^{\alpha}\ \ \textit{for all}\ \ r\in I,$$
or
$$\frac{q_{1r}}{q_{2r}}=\frac{\overline{p}_{1r}}{\overline{p}_{2r}}\Big|\frac{p_{1r}}{p_{2r}}\Big|^{\alpha}\ \ \textit{for all}\ \ r\in I.$$

Theorem 3.2. *Let the matrices $P_r=S\operatorname{diag}\{p_{1r},\dots,p_{n+1,r}\}\,S^{-1}$, $Q_r=T\operatorname{diag}\{q_{1r},\dots,q_{n+1,r}\}\,T^{-1}$, sets of numbers $\big\{\ln\frac{p_{1r}}{p_{n+1,r}},\dots,\ln\frac{p_{nr}}{p_{n+1,r}}\big\}$ and $\big\{\ln\frac{q_{1r}}{q_{n+1,r}},\dots,\ln\frac{q_{nr}}{q_{n+1,r}}\big\}$ are simple, for all $r\in I$. Then for the topological equivalence of systems* (3.1) *and* (3.2) *it is necessary and sufficient existence of such permutations $\mu:I\to I$, $\varrho:(1,\dots,n+1)\to(1,\dots,n+1)$ and complex number α with* $\operatorname{Re}\alpha>-1$, *that either*
$$\frac{q_{\varrho(k)\mu(r)}}{q_{\varrho(n+1)\mu(r)}}=\frac{p_{kr}}{p_{n+1,r}}\Big|\frac{p_{kr}}{p_{n+1,r}}\Big|^{\alpha}\ \ \textit{for all}\ \ r\in I,\ \ k=1,\dots,n,$$
or
$$\frac{q_{\varrho(k)\mu(r)}}{q_{\varrho(n+1)\mu(r)}}=\frac{\overline{p}_{kr}}{\overline{p}_{n+1,r}}\Big|\frac{p_{kr}}{p_{n+1,r}}\Big|^{\alpha}\ \ \textit{for all}\ \ r\in I,\ \ k=1,\dots,n.$$

Theorem 3.3. *From a topological equivalence of systems* (3.1) *and* (3.2) *with the non-Abelian holonomy groups of general situation follows their real holomorphic equivalence.*

Theorem 3.4. *Systems* (3.1) *and* (3.2) *are smooth* (*real holomorphic*) *equivalent if and only if its holonomy groups are conjugated either by linear-fractional transformation or by antiholomorphic linear-fractional transformation for some permutation $\mu:I\to I$.*

Theorem 3.5. *System* (3.1) *is structurally stable if and only if $n=1$, it holonomy group have one independent generator and $|p_{11}p_{21}^{-1}|\neq1$.*

4 Complex autonomous linear differential systems

At first we will consider complex completely solvable [2] (at $m > 1$) nondegenerate [4] linear discrete dynamic systems (L^1) and (L^2), defined by linear maps $A_j w$ for all $w \in \mathbb{C}^n$, $j = 1, \dots, m$, and $B_j w$ for all $w \in \mathbb{C}^n$, $j = 1, \dots, m$, accordingly, where $n > 1$, $1 < m < n - 1$, $A_j \in GL(n, \mathbb{C})$ and $B_j \in GL(n, \mathbb{C})$, $j = 1, \dots, m$, origin O of space $\mathbb{C}^n$ is a unique fixed point of each of these systems.

Definition 4.1. Systems (L^1) and (L^2) we will name ***topologically equivalent*** if there exists the homeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$, translating the layers of the foliation, organised by basis of nondegenerate absolute invariants [4] of system (L^1), into the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (L^2).

In article [6] the criterion of topological equivalence of systems (L^1) and (L^2) of general situation has been obtained. Completely solvable linear discrete dynamic system (L^1) is put in the flow

$$\exp\Big(\sum_{j=1}^{m} z_j \ln A_j\Big) w \ \text{ for all } \ w \in \mathbb{C}^n,$$

defined by the completely solvable autonomous linear differential system

$$dw = \sum_{j=1}^{m} \ln A_j w \, dz_j. \tag{4.1}$$

Therefore on the basis of results of article [6] it is possible to realize topological classification of the autonomous linear differential system (4.1) of general situation.

Notice that topological classification of ordinary system (4.1) (i.e. at $m = 1$) of general situation has been realize in articles [3] and [1].

5 Complex autonomous homogeneous projective matrix Riccati equations

At first we will consider complex completely solvable (at $m > 1$) nondegenerate linear-fractional discrete dynamic systems (PL^1) and (PL^2), defined by linear-fractional maps $A_j v$ for all $v \in \mathbb{C}P^n$, $j = 1, \dots, m$, and $B_j v$ for all $v \in \mathbb{C}P^n$, $j = 1, \dots, m$, accordingly, where $n > 1$, $1 < m < n - 1$, $A_j \in GL(n+1, \mathbb{C})$ and $B_j \in GL(n+1, \mathbb{C})$, $j = 1, \dots, m$, each of these systems has exactly $n + 1$ fixed points on $\mathbb{C}P^n$.

Definition 5.1. Systems (PL^1) and (PL^2) we will name ***topologically equivalent*** if there exists the homeomorphism $h : \mathbb{C}P^n \to \mathbb{C}P^n$, translating the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (PL^1), into the layers of the foliation, organised by basis of nondegenerate absolute invariants of system (PL^2).

In article [6] the criterion of topological equivalence of systems (PL^1) and (PL^2) of general situation has been obtained. Completely solvable linear discrete dynamic system (PL^1) is put in the flow

$$\exp\Big(\sum_{j=1}^{m} z_j \ln A_j\Big) v \ \text{ for all } \ v \in \mathbb{C}P^n,$$

defined by the completely solvable autonomous homogeneous projective matrix Riccati equation

$$dv = \sum_{j=1}^{m} \ln A_j v \, dz_j. \tag{5.1}$$

Therefore on the basis of results of article [6] it is possible to realize topological classification of the autonomous linear differential system (5.1) of general situation.

References

[1] C. Camacho, N. H. Kuiper and J. Palis, The topology of holomorphic flows with singularity. *Inst. Hautes Études Sci. Publ. Math.* No. 48 (1978), 5–38.

[2] I. V. Gayshun, *Completely Solvable Many-Dimensional Differential Equations.* (Russian) Nauka i tekhnika, Minsk, 1983.

[3] N. N. Ladis, Topological equivalence of nonautonomous equations. (Russian) *Differencial'nye Uravnenija* **13** (1977), no. 5, 951–953.

[4] V. Yu. Tyshchenko, Invariants of discrete dynamical systems. (Russian) *Differ. Uravn.* **46** (2010), no. 5, 752–755; translation in *Differ. Equ.* **46** (2010), no. 5, 758–761.

[5] V. Yu. Tyshchenko, *Covering Foliations of Differential Systems.* (Russian) GrSU, Grodno, 2011.

[6] V. Yu. Tyshchenko, On the classification of foliations defined by complex linear and linear-fractional discrete dynamical systems. (Russian) *Vestn. Beloruss. Gos. Univ.*, Ser. 1, Fiz. Mat. Inform. **2012**, No. 3, 125–130.