On One Inverse Problem for the Linear Controlled Neutral Differential Equation

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Let $t_0 < t_1$ be fixed numbers and let $x_0 \in \mathbb{R}^n$ be a fixed vector. By Φ and Ω we denote, respectively, the sets of measurable initial functions $\varphi(t) = (\varphi^1(t), \dots, \varphi^n(t))^T$, $t \in [t_0 - \tau, t_0]$, $\varphi^i(t) \in [-1, 1]$, $i = \overline{1, n}$ and control functions $u(t) = (u^1(t), \dots, u^r(t))^T$, $t \in [t_0, t_1]$, $u^i(t) \in [-1, 1]$, $i = \overline{1, r}$.

To each element $w = (\varphi(t), g(t), u(t)) \in W = \Phi^2 \times \Omega$ we assign the linear neutral differential equation

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \tau) + Du(t), \ \ t \in [t_0, t_1] \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = g(t), \ \ t \in [t_0 - \tau, t_0), \ \ x(t_0) = x_0, \tag{2}$$

where A, B, C, D are given constant matrices with appropriate dimensions.

Definition 1. Let $w = (\varphi(t), g(t), u(t)) \in W$. A function $x(t) = x(t; w) \in \mathbb{R}^n, t \in [t_0 - \tau, t_1]$ is called a solution of differential equation (1) with the initial condition (2) if $x(t)$ satisfies the initial condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere.

The inverse problem: Let $y \in Y = \{y \in \mathbb{R}^n : \ \exists\, w \in W, \ x(t_1; w) = y\}$ be a given vector. Find element $w \in W$ such that the following condition holds $x(t_1; w) = y$. The vector y, as rule, by distinct error is beforehand given.Thus instead of the vector y we have $\widehat{y}$ (so called an observed vector) which is an approximation to the y and, in general, $\widehat{y} \notin Y$.Therefore it is natural to change posed inverse problem by the following approximate problem.

The approximate inverse problem: Find an element $w \in W$ such that the deviation

$$\frac{1}{2}\,|x(t_1; w) - \widehat{y}\,|^2 = \frac{1}{2}\sum_{i=1}^{n} \big[x^i(t_1; w) - \widehat{y}^{\,i}\big]^2$$

takes the minimal value.

It is clear that the approximate inverse problem is equivalent to the following optimization problem:

$$\dot{x}(t)=Ax(t)+Bx(t-\tau)+C\dot{x}(t-\tau)+Du(t),\ \ t\in[t_0,t_1], \tag{3}$$

$$x(t)=\varphi(t),\ \ \dot{x}(t)=g(t),\ \ t\in[t_0-\tau,t_0),\ \ x(t_0)=x_0, \tag{4}$$

$$J(w)=\frac{1}{2}\,|x(t_1;w)-\widehat{y}\,|^2\longrightarrow\min,\ \ w\in W. \tag{5}$$

The problem (3)–(5) is called the optimal control problem corresponding to the inverse problem.

Theorem 1 ([4])**.** *There exists an optimal element $w_0=(\varphi_0(t),g_0(t),u_0(t))$ for the problem* (3)–(5)*, i.e. $J(w_0)=\inf_{w\in W}J(w)$.*

Regularization of the optimal control problem (3)–(5)**.** Now we consider the regularized optimal control problem

$$\dot{x}(t)=Ax+Bx(t-\tau)+C\dot{x}(t-\tau)+Du(t), \tag{6}$$

$$x(t)=\varphi(t),\ \ \dot{x}(t)=g(t),\ \ t\in[t_0-\tau,t_0),\ \ x(t_0)=x_0, \tag{7}$$

$$\begin{aligned}J(w;\delta)=\frac{1}{2}\,|x(t_1;w)-\widehat{y}\,|^2&+\delta_1\int\limits_{t_0}^{t_1}\alpha(t)|\varphi(t-\tau)|^2\,dt\\&+\delta_2\int\limits_{t_0}^{t_1}\alpha(t)|g(t-\tau)|^2\,dt+\delta_3\int\limits_{t_0}^{t_1}|u(t)|^2\,dt\longrightarrow\min,\ \ w\in W,\end{aligned} \tag{8}$$

where $\delta=(\delta_1,\delta_2,\delta_3)$, $\delta_i>0$, $i=1,2,3$ and $\alpha(t)$ is the characteristic function of the interval $[t_0,t_0+\tau]$.

Theorem 2. *For every δ the problem* (6)–(8) *has the unique optimal element $w_\delta=(\varphi_\delta(t),g_\delta(t),u_\delta(t))$ and*

$$\lim_{\delta\to 0}J(w_\delta;\delta)=J(w_0).$$

It is natural that for sufficiently small δ the element w_δ can be considered as an approximate optimal element of the problem (3)–(5) and consequently as an approximate solution of the approximate inverse problem.

Theorem 3. *For the optimality of an element w_δ it suffices to fulfill the conditions:*

$$\psi(t+\tau)B\varphi_\delta(t)-\delta_1|\varphi_\delta(t)|^2=\max_{\varphi\in[-1,1]^n}\big[\psi(t+\tau)B\varphi-\delta_1|\varphi|^2\big],\ \ t\in[t_0-\tau,t_0], \tag{9}$$

$$\psi(t+\tau)Cg_\delta(t)-\delta_2|g_\delta(t)|^2=\max_{g\in[-1,1]^n}\big[\psi(t+\tau)Cg-\delta_2|g|^2\big],\ \ t\in[t_0-\tau,t_0], \tag{10}$$

$$\psi(t)Du_\delta(t)-\delta_3|u_\delta(t)|^2=\max_{u\in[-1,1]^r}\big[\psi(t)Du-\delta_3|u|^2\big],\ \ t\in[t_0,t_1]. \tag{11}$$

Here $\psi(t)$, in general, is discontinuous at points $t_1-k\tau$, $k=1,2,\dots$ and $(\psi(t),\chi(t))$ is a solution of the system

$$\begin{cases}\dot{\chi}(t)=-\psi(t)A-\psi(t+\tau)B,\\ \psi(t)=\chi(t)+C\psi(t+\tau)\end{cases} \tag{12}$$

with the initial condition

$$\psi(t_1)=\chi(t_1)=\widehat{y}-x(t_1;w_\delta),\ \ \psi(t)=0,\ \ t>t_1. \tag{13}$$

Let

$$\psi(t+\tau)B := (\varrho^1(t),\dots,\varrho^n(t)), \quad \psi(t+\tau)C := (\sigma^1(t),\dots,\sigma^n(t)),$$
$$\psi(t)D := (\gamma^1(t),\dots,\gamma^r(t)).$$

Using these notations, from (9)–(11), respectively, it follow

$$\begin{aligned}
\varrho^i(t)\varphi^i_\delta(t) - \delta_1(\varphi^i_\delta(t))^2 &= \max_{\varphi^i\in[-1,1]} \big[\varrho^i(t)\varphi^i - \delta_1(\varphi^i)^2\big], \ \ i=\overline{1,n},\\
\sigma^i(t)g^i_\delta(t) - \delta_2(g^i_\delta(t))^2 &= \max_{g^i\in[-1,1]} \big[\sigma^i(t)g^i - \delta_2(g^i)^2\big], \ \ i=\overline{1,n},\\
\gamma^i(t)u^i_\delta(t) - \delta_3(u^i_\delta(t))^2 &= \max_{u^i\in[-1,1]} \big[\gamma^i(t)u^i - \delta_3(u^i)^2\big], \ \ i=\overline{1,r}.
\end{aligned}$$

From the last relations we get

$$\varphi^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \le -1,\\ \dfrac{\varrho^i(t)}{2\delta_1} & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \in [-1,1],\\ 1 & \text{if } \dfrac{\varrho^i(t)}{2\delta_1} \ge 1, \end{cases} \quad g^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \le -1,\\ \dfrac{\sigma^i(t)}{2\delta_3} & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \in [-1,1],\\ 1 & \text{if } \dfrac{\sigma^i(t)}{2\delta_2} \ge 1, \end{cases}$$
$$u^i_\delta(t) = \begin{cases} -1 & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \le -1,\\ \dfrac{\gamma^i(t)}{2\delta_2} & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \in [-1,1],\\ 1 & \text{if } \dfrac{\gamma^i(t)}{2\delta_3} \ge 1. \end{cases}$$

Iterative process for the approximate solution of the regularization problem (6)–(8)**.** Let $\varphi_1(t)\in\Phi, g_1(t)\in\Phi$ and $u_1(t)\in\Omega$ be starting approximation of the initial functions and the control function. We construct the sequences $\{x_k(t)\}$, $\{\psi_k(t)\}$, $\{\varphi_k(t)\}$, $\{g_k(t)\}$, $\{u_k(t)\}$ by the following iteration process:

1) for given $\varphi_k(t), g_k(t)\in\Phi$ and $u_k(t)\in\Omega$ find $x_k(t)$: the solution of the differential equation

$$\dot x(t) = Ax(t) + Bx(t-\tau) + C\dot x(t-\tau) + Du_k(t), \ \ t\in[t_0,t_1]$$

with the initial condition

$$x(t)=\varphi_k(t), \dot x(t) = g_k(t), \ \ t\in[t-\tau,t_0), \ \ x(t_0)=x_0;$$

2) if a stopping criterion is satisfied stop, stopping criterion can be for example the value of $J(w_k;\delta)$ is less than before given number ε, where $w_k=(\varphi_k(t),g_k(t),u_k(t))$;

3) find $(\psi_k(t),\chi_k(t))$: the solution of the differential equation (12) with the initial condition

$$\psi(t_1)=\chi(t_1)=\widehat y - x(t_1;w_k)\psi(t)=0, \ \ t>t_1;$$

4) put $k := k+1$ and find the next iterates $\varphi_{k+1}(t)$, $g_{k+1}(t)$ and $u_{k+1}(t)$

$$\varphi^i_{k+1}(t)=\begin{cases}-1 & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1}\le -1,\\ \dfrac{\varrho^i_k(t)}{2\delta_1} & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1}\in[-1,1],\\ 1 & \text{if } \dfrac{\varrho^i_k(t)}{2\delta_1}\ge 1,\end{cases}\quad g^i_{k+1}(t)=\begin{cases}-1 & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2}\le -1,\\ \dfrac{\sigma^i_k(t)}{2\delta_2} & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2}\in[-1,1],\\ 1 & \text{if } \dfrac{\sigma^i_k(t)}{2\delta_2}\ge 1,\end{cases}$$

$$u^i_{k+1}(t)=\begin{cases}-1 & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3}\le -1,\\ \dfrac{\gamma^i_k(t)}{2\delta_3} & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3}\in[-1,1],\\ 1 & \text{if } \dfrac{\gamma^i_k(t)}{2\delta_3}\ge 1.\end{cases}$$

Here

$$\psi_k(t+\tau)B := (\varrho^1_k(t),\dots,\varrho^n_k(t)),\quad \psi_k(t+\tau)C := (\sigma^1_k(t),\dots,\sigma^n_k(t)),$$
$$\psi_k(t)D := (\gamma^1_k(t),\dots,\gamma^r_k(t));$$

5) go to 1).

Theorem 4. *The following relations are valid:*

$$\lim_{k\to\infty}\chi_k(t)=\chi_\delta(t),\quad \lim_{k\to\infty}x_k(t)=x_\delta(t)\ \ \textit{uniformly for}\ \ t\in[t_0,t_1],$$
$$\lim_{k\to\infty}\sup_{t\in[t_0,t_1]}\psi_k(t)=\psi_\delta(t),\quad \lim_{k\to\infty}\varphi_k(t)=\varphi_\delta(t),\quad \lim_{k\to\infty}g_k(t)=g_\delta(t)$$

weekly in the space $L_1([t_0-\tau,t_0],\mathbb{R}^n)$, $\lim\limits_{k\to\infty}u_k(t)=u_\delta(t)$ *weekly in the space* $L_1([t_0,t_1],\mathbb{R}^r)$. *Moreover,* $w_\delta=(\varphi_\delta(t),g_\delta(t),u_\delta(t))$ *is the optimal element,* $x_\delta(t)=x(t;w_\delta)$, $(\psi_\delta(t),\chi_\delta(t))$ *is the solution of the equation* (12) *with the initial condition* (13).

Theorems 2–4 are proved on the basis of results obtained in [1–3].

References

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