On a Fundamental Matrix of Linear Homogeneous Differential System with Coefficients of Oscillating Type

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Let
$$G(\varepsilon_0)=\big\{t,\varepsilon:\ 0<\varepsilon<\varepsilon_0,\ -L\varepsilon^{-1}\le t\le L\varepsilon^{-1},\ 0<L<+\infty\big\}.$$

Definition 1. We say that a function $p(t,\varepsilon)$ belongs to the class $S_0(m;\varepsilon_0)$ ($m\in\mathbf{N}\cup\{0\}$) if

1) $p:G(\varepsilon_0)\to\mathbf{C}$;
2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;
3)
$$\frac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k^*(t,\varepsilon)\ \ (0\le k\le m),$$
$$\|p\|_{S_0(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|p_k^*(t,\varepsilon)|<+\infty.$$

Under the slowly varying function we mean the function of the class $S_0(m;\varepsilon_0)$.

Definition 2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F_0(m;\varepsilon_0;\theta)$ ($m\in\mathbf{N}\cup\{0\}$) if this function can be represented as:
$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$
and

1) $f_n(t,\varepsilon)\in S_0(m;\varepsilon_0)$;
2)
$$\|f\|_{F_0(m;\varepsilon_0;\theta)}\overset{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S_0(m;\varepsilon_0)}<+\infty;$$
3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi(t,\varepsilon)\in\mathbf{R}^+$, $\varphi(t,\varepsilon)\in S_0(m;\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Definition 3. We say that a vector-function $a(t,\varepsilon)=\operatorname{colon}(a_1(t,\varepsilon),\dots,a_N(t,\varepsilon))$ belongs to the class $S_1(m;\varepsilon_0)$ if $a_j(t,\varepsilon)\in S_0(m;\varepsilon_0)$ ($j=\overline{1,N}$). We say that a matrix-function $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}(t,\varepsilon)\in S_0(m;\varepsilon_0)$ ($j,k=\overline{1,N}$).

We define the norms:
$$\|a(t,\varepsilon)\|_{S_1(m;\varepsilon_0)}=\max_{1\le j\le N}\|a_j(t,\varepsilon)\|_{S_0(m;\varepsilon_0)},$$
$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}=\max_{1\le j\le N}\sum_{k=1}^{N}\|a_{jk}(t,\varepsilon)\|_{S_0(m;\varepsilon_0)}.$$

Definition 4. We say that a vector-function $b(t,\varepsilon,\theta) = \operatorname{colon}(b_1(t,\varepsilon,\theta),\dots,b_N(t,\varepsilon,\theta))$ belongs to the class $F_1(m;\varepsilon_0;\theta)$ if $b_j(t,\varepsilon,\theta) \in F_0(m;\varepsilon_0;\theta)$ $(j=\overline{1,N})$. We say that a matrix-function $B(t,\varepsilon,\theta) = (b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ if $b_{jk}(t,\varepsilon,\theta) \in F_0(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

We define the norms:

$$\|b(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)} = \max_{1\le j\le N} \|b_j(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)},$$

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N} \sum_{k=1}^{N} \|b_{jk}(t,\varepsilon,\theta)\|_{F_0(m;\varepsilon_0;\theta)}.$$

Thus, the matrix $B(t,\varepsilon,\theta)$ has a kind

$$B(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} B_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

where $B_n(t,\varepsilon) \in S_2(m;\varepsilon_0)$, and

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} \le \sum_{n=-\infty}^{\infty} \|B_n(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}.$$

It is easy to obtain that if $A, B \in F_2(m;\varepsilon_0;\theta)$, then $AB \in F_2(m;\varepsilon;\theta)$, and

$$\|AB\|_{F_2(m;\varepsilon_0;\theta)} \le 2^m \|A\|_{F_2(m;\varepsilon_0;\theta)} \cdot \|B\|_{F_2(m;\varepsilon_0;\theta)}.$$

For $A(t,\varepsilon,\theta) \in F_2(m;\varepsilon_0;\theta)$ we denote

$$\Gamma_n[A] = \frac{1}{2\pi}\int_0^{2\pi} A(t,\varepsilon,\theta)\exp(-in\,\theta)\,d\theta \quad (n\in\mathbf{Z}).$$

We consider the next system of differential equations

$$\frac{dx}{dt} = \big(\Lambda(t,\varepsilon) + \varepsilon P(t,\varepsilon,\theta)\big)x, \tag{1}$$

where $\varepsilon\in(0,\varepsilon_0)$, $\Lambda(t,\varepsilon) = \operatorname{diag}(\lambda_1(t,\varepsilon),\dots,\lambda_N(t,\varepsilon)) \in S_2(m;\varepsilon_0)$, $P(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$.

We study the problem about the structure of fundamental matrix of the system (1).

Consider the linear homogeneous system

$$\frac{dx}{dt} = \varepsilon A(t,\varepsilon)x, \tag{2}$$

where $\varepsilon\in(0,\varepsilon_0)$, $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}} \in S_2(m;\varepsilon_0)$. Then there exists a matrizant $X(t,\varepsilon)$ of the system (2).

Lemma 1. *If $X(t,\varepsilon)$ is the matrizant of the system* (2), *then $X(t,\varepsilon)$, $X^{-1}(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$.*

Lemma 2. *Let we have the matrix equation*

$$\frac{dX}{dt} = \varepsilon A(t,\varepsilon,\theta), \tag{3}$$

where $\varepsilon\in(0,\varepsilon_0)$, $A(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$. Then there exists a solution $X(t,\varepsilon,\theta)$ of the equation (3) *which belongs to the class $F_2(m;\varepsilon_0;\theta)$, and there exists $K\in(0,+\infty)$ which does not depend on $A(t,\varepsilon,\theta)$ such that*

$$\|X(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} \le K\|A(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)}.$$

Theorem 1. *Let the system* (1) *be such that*

$$\inf_{G(\varepsilon_0)} \big|\mathrm{Re}\,(\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)\big| \geq \gamma > 0 \ \ (j\neq k),$$

and $m\geq 1$. *Then there exists* $\varepsilon^*\in(0,\varepsilon_0)$ *such that for all* $\varepsilon\in(0,\varepsilon^*)$ *there exists a fundamental matrix* $X^{(1)}(t,\varepsilon,\theta)$ *of the system* (1) *which has a kind*

$$X^{(1)}(t,\varepsilon,\theta)=R^{(1)}(t,\varepsilon,\theta)\exp\bigg(\int\limits_0^t \Lambda^{(1)}(\tau,\varepsilon)\,d\tau\bigg),$$

where $R^{(1)}(t,\varepsilon,\theta)\in F_2(m-1;\varepsilon^*;\theta)$, $\Lambda^{(1)}(t,\varepsilon)$ *– the diagonal matrix, belonging to the class* $S(m-1;\varepsilon^*)$.

Theorem 2. *Let the system* (1) *be such that*

$$\Lambda(t,\varepsilon)=i\varphi(t,\varepsilon)J,$$

where $\varphi(t,\varepsilon)$ *is function in the Definition* 2, $J=\mathrm{diag}\,(n_1,\dots,n_N)$, $n_j\in\mathbf{Z}$ $(j=\overline{1,N})$, *and* $m\geq 1$. *Then there exists* $\varepsilon^{**}\in(0,\varepsilon_0)$ *such that for all* $\varepsilon\in(0,\varepsilon^{**})$ *there exists a fundamental matrix* $X^{(2)}(t,\varepsilon,\theta)$ *of the system* (1) *which has a kind:*

$$X^{(2)}(t,\varepsilon,\theta(t,\varepsilon))=\exp(i\theta(t,\varepsilon)J)R^{(2)}(t,\varepsilon,\theta(t,\varepsilon)),$$

where $R^{(2)}(t,\varepsilon,\theta(t,\varepsilon))\in F_2(m-1;\varepsilon^{**};\theta)$.

Proof. We make a substitution in the system (1)

$$x=\exp(i\theta(t,\varepsilon)J)y, \tag{4}$$

where y is a new unknown N-dimensional vector. We obtain

$$\frac{dy}{dt}=\varepsilon Q(t,\varepsilon,\theta)y, \tag{5}$$

where $Q(t,\varepsilon,\theta)=\exp(-i\theta(t,\varepsilon)J)P(t,\varepsilon,\theta)\exp(i\theta(t,\varepsilon)J)$ belongs to the class $F_2(m;\varepsilon_0;\theta)$.

Now in the system (5) we make the substitution

$$y=(E+\varepsilon\Phi(t,\varepsilon,\theta))z, \tag{6}$$

where the matrix Φ is defined from the equation

$$\varphi(t,\varepsilon)\,\frac{\partial\Phi}{\partial\theta}=Q(t,\varepsilon,\theta)-U(t,\varepsilon),$$

in which $U(t,\varepsilon)=\Gamma_0[Q(t,\varepsilon,\theta)]$. Then

$$\Phi(t,\varepsilon,\theta)=\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{\Gamma_n[Q(t,\varepsilon,\theta)]}{in\,\varphi(t,\varepsilon)}\,\exp(in\,\theta)\in F_2(m;\varepsilon_0;\theta).$$

As a result of the substitution (6) we obtain

$$\frac{dz}{dt}=\varepsilon\big(U(t,\varepsilon)+\varepsilon V(t,\varepsilon,\theta)\big)z, \tag{7}$$

where the matrix V is defined from the equation

$$(E+\varepsilon\Phi(t,\varepsilon,\theta))V=Q(t,\varepsilon,\theta)\Phi(t,\varepsilon,\theta)-\Phi(t,\varepsilon,\theta)U(t,\varepsilon)-\frac{1}{\varepsilon}\,\frac{\partial\Phi(t,\varepsilon,\theta)}{\partial t}\,. \tag{8}$$

The matrix $\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial t}$ belongs to the class $F_2(m-1;\varepsilon_0;\theta)$, then there exists $\varepsilon_2\in(0,\varepsilon_0)$ such that for all $\varepsilon\in(0,\varepsilon_2)$ the equation (8) is solved with respect to V, and $V(t,\varepsilon,\theta)$ belongs to the class $F_2(m-1;\varepsilon_2;\theta_0)$.

Together with the system (7) we consider the truncated system

$$\frac{dz^{(0)}}{dt}=\varepsilon U(t,\varepsilon)z^{(0)}. \tag{9}$$

Continuity of the matrix $U(t,\varepsilon)$ with respect to t for all $\varepsilon\in(0,\varepsilon_0)$ guarantees the existence of the matrizant $Z^{(0)}(t,\varepsilon)$ of the system (9), and by virtue the Lemma 1 $Z^{(0)}(t,\varepsilon)$, $(Z^{(0)}(t,\varepsilon))^{-1}$ belong to the class $S_2(m-1;\varepsilon_0)$.

We make in the system (7) the substitution

$$z=Z^{(0)}(t,\varepsilon)\xi, \tag{10}$$

where ξ – the new unknown vector. We obtain

$$\frac{d\xi}{dt}=\varepsilon^2W(t,\varepsilon,\theta)\xi, \tag{11}$$

where $W=(Z^{(0)}(t,\varepsilon))^{-1}V(t,\varepsilon,\theta)Z^{(0)}(t,\varepsilon))\in F_2(m-1;\varepsilon_2;\theta)$.

Now we show that there exists a substitution

$$\xi=(E+\varepsilon\Psi(t,\varepsilon,\theta))\eta, \tag{12}$$

where $\Psi\in F_2(m-1;\varepsilon_3;\theta)$ $(\varepsilon_3\in(0,\varepsilon_2))$, which leads the system (11) to the system

$$\frac{d\eta}{dt}=O\eta, \tag{13}$$

where O – the null $(N\times N)$-matrix. Really, we define the matrix Ψ from the equation

$$\frac{d\Psi}{dt}=\varepsilon W(t,\varepsilon,\theta)+\varepsilon^2W(t,\varepsilon,\theta)\Psi. \tag{14}$$

Consider the truncated equation

$$\frac{d\Psi^{(0)}}{dt}=\varepsilon W(t,\varepsilon,\theta).$$

By virtue of Lemma 2 this equation has a solution $\Psi^{(0)}(t,\varepsilon,\theta)\in F_2(m-1;\varepsilon_2;\theta)$.

We construct the process of successive approximations, used as initial approximation $\Psi^{(0)}(t,\varepsilon,\theta)$, and the subsequent approximations defining as solutions from the class $F_2(m-1;\varepsilon_2;\theta)$ of the matrix-equations

$$\frac{d\Psi^{(k+1)}}{dt}=\varepsilon W(t,\varepsilon,\theta)+\varepsilon^2W(t,\varepsilon,\theta)\Psi^{(k)},\ \ k=0,1,2,\dots\,. \tag{15}$$

Each of these solutions exists by virtue of Lemma 2. Then we have

$$\frac{d(\Psi^{(k+1)}-\Psi^{(k)})}{dt}=\varepsilon^2W(t,\varepsilon,\theta)(\Psi^{(k)}-\Psi^{(k-1)}),\ \ k=1,2,\dots\,.$$

By virtue of Lemma 2 and unequality (2) we obtain

$$\|\Psi^{(k+1)} - \Psi^{(k)}\|_{F_2(m-1;\varepsilon_2;\theta)} \le \varepsilon 2^{m-1} K \|\Psi^{(k)} - \Psi^{(k-1)}\|_{F_2(m-1;\varepsilon_2;\theta)}, \quad k = 1, 2, \dots$$

(K is defined in the Lemma 2), therefore the convergence of the process (15) is guaranteed by the unequality $0 < \varepsilon < \varepsilon_3$, where $\varepsilon_3 2^{m-1} K < 1$. As a result of the process (15) we obtain the solution $\Psi(t, \varepsilon, \theta)$, belonging to the class $F_2(m-1; \varepsilon_3; \theta)$, of the equation (14).

The matrizant of the system (13) is E. Thus, by virtue of (4), (6), (10), (12) we obtain that the fundamental matrix of the system (1) has a kind:

$$X^{(2)}(t, \varepsilon, \theta) = \exp(i\theta(t, \varepsilon)J)(E + \varepsilon\Phi(t, \varepsilon, \theta))Z^{(0)}(t, \varepsilon)(E + \varepsilon\Psi(t, \varepsilon, \theta)),$$

and the Theorem 2 is proved. $\square$

Remark. In the sense of the condition of Theorem 2 we say that we have a resonance case.