Necessary Conditions of Optimality for the Optimal Control Problem with Several Delays and the Continuous Initial Condition

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Let $O\subset\mathbb{R}^n$ be an open set and $U\subset\mathbb{R}^r$ be a convex compact set. Let $h_{i2}>h_{i1}>0$, $i=\overline{1,s}$ and $\theta_k>\cdots>\theta_1>0$ be given numbers and n-dimensional function $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)$, $(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)\in I\times O^{1+s}\times U^{1+k}$ satisfies the following conditions: for almost all fixed $t\in I=[a,b]$ the function $f(t,\,\cdot\,):I\times O^{1+s}\times U^{1+k}\to\mathbb{R}^n$ is continuous and continuously differentiable in $(x,x_1,\dots,x_s,u,u_1,\dots,u_k)\in O^{1+s}\times U^{1+k}$; for each fixed $(x,x_1,\dots,x_s,u,u_1,\dots,u_k)\in O^{1+s}\times U^{1+k}$, the function $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)$ and the matrices $f_x(t,\,\cdot\,)$, $f_{x_i}(t,\,\cdot\,)$, $i=\overline{1,s}$ and $f_u(t,\,\cdot\,)$, $f_{u_i}(t,\,\cdot\,)$, $i=\overline{1,k}$ are measurable on I; for any compact set $K\subset O$ there exists a function $m_K(t)\in L_1(I,[0,\infty))$ such that

$$\big|f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)\big| + |f_x(t,x,\,\cdot\,)|+\sum_{i=1}^{s}|f_{x_i}(t,x,\,\cdot\,)|+|f_u(t,x,\,\cdot\,)|+\sum_{i=1}^{k}|f_{u_i}(t,x,\,\cdot\,)|\le m_K(t)$$

for all $(x,x_1,\dots,x_s,u,u_1,\dots,u_k)\in K^{1+s}\times U^{1+k}$ and for almost all $t\in I$.

Furthermore, let Φ be the set of continuous functions $\varphi(t)\in N$, $t\in I_1=[\widehat{\tau},b]$, where $\widehat{\tau}=a-\max\{h_{12},\dots,h_{s2}\}$, $N\subset O$ is a convex compact set; Ω is the set of measurable functions $u(t)\in U$, $t\in I_2=[a-\theta_k,b]$.

To each element $v=(t_0,t_1,\tau_1,\dots,\tau_s,\varphi,u)\in A=I\times I\times[h_{11},h_{12}]\times\cdots\times[h_{s1},h_{s2}]\times\Phi\times\Omega$ on the interval $[t_0,t_1]$ we assign the delay controlled functional differential equation

$$\dot{x}(t)=f\Big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t),u(t-\theta_1),\dots,u(t-\theta_k)\Big) \tag{1}$$

with the continuous initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat{\tau},t_0]. \tag{2}$$

The condition (2) is called continuous because always $x(t_0)=\varphi(t_0)$.

Definition 1. Let $\nu=(t_0,t_1,\tau_1,\dots,\tau_s,\varphi,u)\in A$. A function $x(t)=x(t;\nu)\in O$, $t\in[\widehat{\tau},t_1]$, $t_1\in(t_0,b]$ is called a solution of equation (1) with the continuous initial condition (2), or the solution corresponding to ν and defined on the interval $[\widehat{\tau},t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let the scalar-valued functions $q^i(t_0,t_1,\tau_1,\dots,\tau_s,x_0,x_1)$, $i=\overline{0,l}$ be continuously differentiable on $I^2\times[h_{11},h_{12}]\times\cdots\times[h_{s1},h_{s2}]\times O^2$.

Definition 2. An element $\nu=(t_0,t_1,\tau_1,\dots,\tau_s,\varphi,u)\in A$ is said to be admissible if the corresponding solution $x(t)=x(t;\nu)$ satisfies the boundary conditions

$$q^i\big(t_0,t_1,\tau_1,\dots,\tau_s,\varphi(t_0),x(t_1)\big)=0,\ \ i=\overline{1,l}. \tag{3}$$

Denote by A_0 the set of admissible elements.

Definition 3. An element $\nu_0=(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in A_0$ is said to be optimal if for an arbitrary element $\nu\in A_0$ the inequality

$$q^0\big(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0(t_{00}),x_0(t_{10})\big)\le q^0\big(t_0,t_1,\tau_1,\dots,\tau_s,\varphi(t_0),x(t_1)\big) \tag{4}$$

holds. Here $x_0(t)=x(t;\nu_0)$ and $x(t)=x(t;\nu)$.

The problem (1)–(4) is called the optimal control problem with the continuous initial condition.

Theorem 1. *Let ν_0 be an optimal element with $t_{00},t_{10}\in(a,b)$ and the following conditions hold:*

1) *the function $\varphi_0(t)$ is absolutely continuous and $\dot\varphi_0(t)$ is bounded;*

2) *the function*
$$f_0(w)=f\big(w,u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k)\big),$$
where $w=(t,x,x_1,\dots,x_s)\in I\times O^{1+s}$ is bounded on $I\times O^{1+s}$;

3) *there exists the finite limits*
$$\lim_{t\to t_{00}-}\dot\varphi_0(t)=\dot\varphi_0^-,\quad \lim_{w\to w_0}f_0(w)=f_0^-,\ \ w\in(a,t_{00}]\times O^{1+s},$$
where
$$w_0=\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0})\big);$$

4) *there exists the finite limit*
$$\lim_{w\to w_1}f_0(w)=f_1^-,\ \ w\in(t_{00},t_{10}]\times O^{1+s},$$
$$w_1=\big(t_{10},x_0(t_{10}),x_0(t_{10}-\tau_{10}),\dots,x_0(t_{10}-\tau_{s0})\big).$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\ne0$, with $\pi_0\le0$, and a solution $\psi(t)=(\psi_1(t),\dots,\psi_n(t))$ of the equation

$$\dot\psi(t)=-\psi(t)f_{0x}[t]-\sum_{i=1}^{s}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}],\ \ t\in[t_{00},t_{10}],\ \ \psi(t)=0,\ \ t>t_{10}, \tag{5}$$

where
$$f_{0x}[t]=f_{0x}\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0})\big),$$
such that the following conditions hold;

5) *the conditions for the moments t_{00} and t_{10}:*
$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\dot\varphi_0^-\ge\psi(t_{00})f^-,\ \ \pi Q_{0t_1}\ge-\psi(t_{10})f_1^-,$$
where
$$Q_{0t_0}=\frac{\partial}{\partial t_0}\,Q\big(t_{00},t_{10},\tau_{10},\dots,\tau_{s0},\varphi_0(t_{00}),x_0(t_{10})\big),\ \ Q=(q^0,\dots,q^l)^T;$$

6) *the conditions for the delays* τ_{i0}, $i=\overline{1,s}$,

$$\pi Q_{0\tau_i}=\int\limits_{t_{00}}^{t_{10}}\psi(t)f_{0x_i}[t]\,\dot{x}_0(t-\tau_{i0})\,dt,\ \ i=\overline{1,s};$$

7) *the maximum principle for the initial function* $\varphi_0(t)$,

$$[Q_{0x_0}+\psi(t_{00})]\varphi_0(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi_0(t)\,dt$$
$$=\max_{\varphi(t)\in\Phi}\bigg\{[Q_{0x_0}+\psi(t_{00})]\varphi(t_{00})+\sum_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}\psi(t+\tau_{i0})f_{0x_i}[t+\tau_{i0}]\varphi(t)\,dt\bigg\};$$

8) *the linearized integral maximum principle for the control function* $u_0(t)$,

$$\int\limits_{t_{00}}^{t_{10}}\psi(t)\Big[f_{0u}[t]u_0(t)+\sum_{i=1}^{k}f_{0u_i}[t]u_0(t-\theta_i)\Big]\,dt$$
$$=\max_{u(t)\in\Omega}\int\limits_{t_{00}}^{t_{10}}\psi(t)\Big[f_{0u}[t]u(t)+\sum_{i=1}^{k}f_{0u_i}[t]u(t-\theta_i)\Big]\,dt;$$

9) *the condition for the function* $\psi(t)$

$$\psi(t_{10})=\pi Q_{0x_1}.$$

Theorem 2. *Let* ν_0 *be an optimal element with* $t_{00},t_{10}\in(a,b)$ *and the conditions* 1), 2) *of Theorem* 1 *hold. Moreover, there exists the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0}f_0(w)=f_0^+,\ \ w\in[t_{00},b)\times O^{1+s},$$
$$\lim_{w\to w_1}f_0(w)=f_1^+,\ \ w\in[t_{10},b)\times O^{1+s}.$$

Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution* $\psi=(\psi_1(t),\dots,\psi_n(t))$ *of the equation* (5) *such that the conditions* 6)–9) *hold. Moreover,*

$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\dot{\varphi}_0^+\le\psi(t_{00})f_0^+,\ \ \pi Q_{0t_1}\le-\psi(t_{10})f_1^+,$$

Theorem 3. *Let* ν_0 *be an optimal element with* $t_{00},t_{10}\in(a,b)$ *and the following conditions hold: the function* $\varphi_0(t)$ *is continuously differentiable; the function* $f(t,x,x_1,\dots,x_s,u,u_1,\dots,u_k)$ *is continuous; the function* $f(t,x,x_1,\dots,x_s,u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k))$ *is continuous at points* t_{00},t_{10}. *Then there exist a vector* $\pi=(\pi_0,\dots,\pi_l)\neq 0$, *with* $\pi_0\le 0$, *and a solution* $\psi=(\psi_1(t),\dots,\psi_n(t))$ *of the equation* (5) *such that the conditions* 6)–9) *hold. Moreover,*

$$\pi Q_{0t_0}+(\pi Q_{0x_0}+\psi(t_{00}))\varphi_0(t_{00})=\psi(t_{00})f_0[t_{00}],\ \ \pi Q_{0t_1}=-\psi(t_{10})f_0[t_{10}],$$

where

$$f_0[t]=f\Big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0}),u_0(t),u_0(t-\theta_1),\dots,u_0(t-\theta_k)\Big).$$

Theorem 3 is a corollary to Theorems 1 and 2. On the basis of variation formulas [2,3] Theorems 1, 2 are proved by the scheme given in [1,4].

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