Asymptotic Behavior of Solutions for One Class of Third Order Nonlinear Differential Equations

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Consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^\sigma, \tag{1}$$

where $\alpha_0 \in \{-1;1\}$, $p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\sigma \in \mathbb{R}$, $\infty < a < \omega \leq +\infty$. It belongs to the equations class of the form

$$y''' = \alpha_0 p(t)\, L(y), \tag{2}$$

where $\alpha_0 \in \{-1;1\}$,$p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\infty < a < \omega \leq +\infty$, function L continuous and positive in a one-sided neighborhood Δ_{Y_0} points Y_0 (Y_0 equals either 0 or $\pm\infty$).

For equations of the form (2) in the works of A. Stekhun and V. Evtukhov [4, 9] there was investigated the question of the existence and asymptotic behavior when $t \to \omega$ of the endangered and unlimited solutions. The method of studying the equation of the form (2) assumed the presence of significant linearity of the function $L(y)$. In the equation (1) the function $L(y) = y\,|\ln |y||^\sigma$ is in some sense close to linear and requires improvements in research methods.

For second order equations of the form (1) in the works of V. Evtukhov and M. Jaber [1,3] there was investigated the question of the existence and asymptotic behavior, when $t \uparrow \omega$ all, so-called $P_\omega(\lambda_0)$-solution.

Solution y of the equation (1), specified on the interval $[t_y, w) \subset [a,\omega)$ is said to be $P_\omega(\lambda_0)$-solution, if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} \ \ 0, \\ \text{or} \ \ \pm\infty, \end{cases} \quad (k=0,1,2), \quad \lim_{t\uparrow\omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0 \tag{3}$$

Earlier in the articles [6–8] were obtained the results in the case, when $\lambda_0 \in R \setminus \{0,-1,\frac{1}{2}\}$. The goal of the work to establish existence conditions for the equation (1) of $P_\omega(\pm\infty)$-solutions and also asymptotic representations, when $t \uparrow \omega$ such solutions and their derivative to the second order.

We introduce the necessary notation for further, assuming

$$q(t) = p(t)\pi_\omega^3(t)\,|\ln \pi_\omega^2(t)|^\sigma, \quad Q(t) = \int_a^t p(\tau)\pi_\omega^2(\tau)\,|\ln \pi_\omega^2(t)|^\sigma\, d\tau,$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{if} \ \ w = +\infty, \\ t-\omega, & \text{if} \ \ w < +\infty. \end{cases}$$

Theorem 1. *For the existence of $P_\omega(\pm\infty)$-solutions of (1), it is necessary and sufficient the conditions*
$$\lim_{t\uparrow\omega} q(t)=0, \quad \lim_{t\uparrow\omega} Q(t)=\infty \tag{4}$$
to be satisfied. Moreover, for each such solution the following asymptotic representations, when $t\uparrow\omega$
$$\begin{gathered}\ln|y(t)|=\ln\pi_\omega^2(t)+\frac{\alpha_0}{2}\,Q(t)[1+o(1)],\\ \ln|y'(t)|=\ln|\pi_\omega(t)|+\frac{\alpha_0}{2}\,Q(t)[1+o(1)],\quad \ln|y''(t)|=\frac{\alpha_0}{2}\,Q(t)[1+o(1)]\end{gathered} \tag{5}$$
take place.

Indeed, if $y:[t_y,\omega[\,\to\mathbb{R}$ is a $P_\omega(\pm\infty)$-solution of the equation (1), then the conditions (3) are met and the following limit relations are true:
$$\lim_{t\uparrow\omega}\frac{y'''(t)\pi_\omega(t)}{y''(t)}=0,\quad \lim_{t\uparrow\omega}\frac{y''(t)\pi_\omega(t)}{y'(t)}=1, \tag{6}$$
$$\lim_{t\uparrow\omega}\frac{y''(t)\pi_\omega^2(t)}{y(t)}=2,\quad \lim_{t\uparrow\omega}\frac{y'(t)\pi_\omega(t)}{y(t)}=2. \tag{7}$$
Without loss of generality, we can assume that $y''(t)$, $y'(t)$, $\ln|y(t)|$ are non-zero when $t\in[t_y,\omega[$. Therefore, considering the limiting relations (7) and formulas
$$y(t)\sim\frac{1}{2}\,\pi_\omega^2(t)\,y''(t),\quad \ln|y(t)|\sim\ln\pi_\omega^2(t)\ \text{ when }\ t\uparrow\omega,$$
from equation (1) we get
$$y'''(t)=\alpha_0 p(t)\,\frac{\pi_\omega^2(t)}{2}\,|\ln\pi_\omega^2(t)|^\sigma y''(t)\,[1+o(1)]. \tag{8}$$
Hence, in view of the first of limiting relations (6), it follows that
$$p(t)\pi_\omega^3(t)\,|\ln\pi_\omega^2(t)|^\sigma\longrightarrow 0\ \text{ when }\ t\uparrow\omega,$$
that is, the first of the conditions (4) of the theorem is satisfied. Dividing now (8) by $y''(t)$ and integrating obtained relation on the interval from t_y to t, come to a conclusion considering the first from conditions (4) that $\int\limits_{t_y}^{\omega} p(t)\pi_\omega^2(t)|\ln\pi_\omega^2(t)|^\sigma\,dt=\infty$ and when $t\uparrow w$ the asymptotic relation
$$\ln|y''(t)|=\frac{\alpha_0}{2}\int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,|\ln\pi_\omega^2(\tau)|^\sigma\,d\tau\,[1+o(1)]$$
take place, that is, the second of the theorem conditions (4) is met and the third of the asymptotic relations (5).

The validity of the first and second asymptotic representations (5) directly follows from the third, considering that $y(t)\sim\frac{1}{2}\,\pi_\omega^2(t)\,y''(t)$ and $y'(t)\sim\pi_\omega(t)\,y''(t)$ when $t\uparrow\omega$.

Assuming that conditions (4) are met, we reduce equation (1) using transformations
$$\begin{gathered}\ln|y(t)|=\ln\pi_\omega^2(\tau)[1+v_1(\tau)],\quad \frac{y'(t)}{y(t)}=\frac{2[1+v_2(\tau)]}{\pi_\omega(t)}\,,\\ \Big(\frac{y'(t)}{y(t)}\Big)'=\frac{-2\,[1+v_3(\tau)]}{\pi_\omega^2(t)}\,,\quad \tau=\beta\ln|\pi_w(t)|,\quad \beta=\begin{cases}1, & \text{when }\ w=+\infty,\\ -1, & \text{when }\ w<+\infty,\end{cases}\end{gathered} \tag{9}$$

to a system of differential equations

$$\begin{cases} v_1' = \dfrac{1}{\tau}\,[v_2 - v_1], \\ v_2' = \beta[v_2 - v_3], \\ v_3' = \beta\big[f(\tau) + \sigma f(\tau)v_1 + 6v_2 - 4v_3 + V(\tau, v_1, v_2, v_3)\big], \end{cases} \tag{10}$$

in which

$$f(\tau) = f(\tau(t)) = \alpha_0 q(t), \quad V(\tau, v_1, v_2, v_3) = 12v_2^2 + 4v_2^3 - 6v_2v_3 + f(\tau)\big[\,|1+v_1|^\sigma - 1 - \sigma v_1\big].$$

For the system (10) all the conditions of the Theorem 2.6 from the work [2] are satisfied. According to that theorem the system (10) has at least one solution $(v_1, v_2, v_3) : [\tau_1, +\infty) \to R^3 (\tau_1 \geq \tau_0)$, converges to zero when $\tau \to +\infty$, to which, due to replacements (9), matches the solution $y(t)$ of the differential equation (1), allowing the asymptotic representations (5) when $t \uparrow \omega$.

Theorem 2. *Let the function $p : [a, \omega) \to (0, +\infty)$ be continuously differentiable and along with the conditions* (4) *the following conditions*

$$\int_a^\omega |q'(t)|\,dt < +\infty, \quad \int_a^\omega \frac{q^2(t)}{|\pi_\omega(t)|}\,dt < +\infty, \quad \int_a^\omega \frac{q(t)|Q(t)|}{\pi_\omega(t)\ln|\pi_\omega(t)|}\,dt < +\infty$$

hold. Then for any $c \neq 0$ equation (1) *has $P_\omega(\pm\infty)$-solution. Furthermore, for every such solution the following asymptotic representations when $t \to w$*

$$y(t) = \pi_\omega^2(t)\,e^{\alpha_0 Q(t)}[c + o(1)],$$
$$y'(t) = \pi_\omega(t)\,e^{\alpha_0 Q(t)}[2c + o(1)], \quad y''(t) = e^{\alpha_0 Q(t)}[2c + o(1)]$$

take place.

Let present a corollary of these theorems, when $\sigma = 0$, i.e. for the following linear differential equation

$$y''' = \alpha_0 p(t) y, \tag{11}$$

where $\alpha_0 \in \{-1; 1\}$, $\sigma \in \mathbb{R}$, $p : [a, w) \to (0, +\infty)$ – continuous function; $a < w \leq +\infty$.

Corollary 1. *For the existence of $P_\omega(\pm\infty)$-solutions of* (11)*, it is necessary and sufficient the conditions*

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^3(t) = 0, \quad \lim_{t\uparrow\omega} \int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau = \infty \tag{12}$$

to be fulfilled. Furthermore, for any such solution the following asymptotic representations, when $t \uparrow \omega$

$$\ln|y(t)| = \ln \pi_\omega^2(t) + \frac{\alpha_0}{2} \int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)],$$

$$\ln|y'(t)| = \ln|\pi_\omega(t)| + \frac{\alpha_0}{2} \int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)],$$

$$\ln|y''(t)| = \frac{\alpha_0}{2} \int_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,[1 + o(1)]$$

take place.

Corollary 2. *Let the function $p : [a,\omega) \to (0,+\infty)$ be continuously-differentiable and along with the conditions* (12) *the following conditions*

$$\int\limits_a^\omega \big|(p(t)\pi_\omega^3(t))'\big|\,dt < +\infty, \quad \int\limits_a^\omega p^2(t)|\pi_\omega^5(t)|\,dt < +\infty,$$

$$\int\limits_a^\omega \frac{p(t)\pi_\omega^2(t)}{\ln|\pi_\omega(t)|} \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\,dt < +\infty$$

hold. Then for any $c \neq 0$ equation (11) *has $P_\omega(\pm\infty)$-solution. Furthermore, for any such solution the following asymptotic representations, when $t \to w$:*

$$y(t) = \pi_\omega^2(t)\exp\bigg(\alpha_0 \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[c + o(1)],$$

$$y'(t) = \pi_\omega(t)\exp\bigg(\alpha_0 \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[2c + o(1)],$$

$$y''(t) = \exp\bigg(\alpha_0 \int\limits_a^t p(\tau)\pi_\omega^2(\tau)\,d\tau\bigg)[2c + o(1)]$$

take place.

The obtained asymptotes are consistent with the already known results for linear differential equations (see [5, Chapter 1]).

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