On Solution of Some Non-Linear Integral Boundary Value Problem

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We study the non-linear integral boundary value problem

$$\frac{dx(t)}{dt} = f\Big(t, x(t), \frac{dx(t)}{dt}\Big), \;\; t \in [a,b], \tag{1}$$

$$g\bigg(x(a), x(b), \int\limits_a^b h(s, x(s))\,ds\bigg) = d. \tag{2}$$

We suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ is continuous function defined on bounded sets $D \subset \mathbb{R}^n$, $D^1 \subset \mathbb{R}^n$ (domain $D := D_\rho$ will be concretized later, see (8), D^1 is given) and $d \in \mathbb{R}^n$ is a given vector. Moreover, $f, g : D \times D \times D_2 \to \mathbb{R}^n$ and $h : [a,b] \times D \to \mathbb{R}^n$ are Lipschitzian in the following form

$$\big|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})\big| \le K_1|u-\widetilde{u}| + K_2|v-\widetilde{v}|, \tag{3}$$

$$\big|g(u,w,p) - g(\widetilde{u},\widetilde{w},\widetilde{p})\big| \le K_3|u-\widetilde{u}| + K_4|w-\widetilde{w}| + K_5|p-\widetilde{p}|, \tag{4}$$

$$\big|h(t,u) - h(t,\widetilde{u})\big| \le K_6|u-\widetilde{u}| \tag{5}$$

for any $t \in [a,b]$ fixed, all $\{u,\widetilde{u}\} \subset D$, $\{v,\widetilde{v}\} \subset D^1$, $\{w,\widetilde{w}\} \subset D$, $\{p,\widetilde{p}\} \subset D_2$, where $D_2 := \Big\{ \int\limits_a^b h(t,x(t))\,dt : \; t \in [a,b], \; x \in D \Big\}$ and $K_1 - K_6$ are non-negative square matrices of dimension n. The inequalities between vectors are understood componentwise. A similar convention is adopted for the operations "absolute value", "max", "min". The symbol I_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K.

By the solution of the problem (1), (2) we understand a continuously differentiable function with property (2) satisfying (1) on $[a,b]$.

In the sequel, we will use an approach that was suggested in [1]. We fix certain bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the solutions x of the given problem with property $x(a) \in D_a$ and $x(b) \in D_b$. Instead of the non-local boundary value problem (1), (2), we consider the parameterized family of two-point "model-type" problems with simple separated conditions

$$\frac{dx(t)}{dt} = f\Big(t, x(t), \frac{dx(t)}{dt}\Big), \;\; t \in [a,b], \tag{6}$$

$$x(a) = z, \quad x(b) = \eta, \tag{7}$$

where $z = (z_1, z_2, \dots, z_n)$, $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ are considered as parameters.

If $z \in \mathbb{R}^n$ and ρ is a vector with non-negative components, $B(z,\rho) := \{\xi \in \mathbb{R}^n : \; |\xi - z| \le \rho\}$ stands for the componentwise ρ neighbourhood of z. For given two bounded connected sets $D_a \subset$

$\mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set $D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a,\ \eta \in D_b,\ \theta \in [0,1]$ and its componentwise ρ-neighbourhood by putting

$$D = D_\rho := B(D_{a,b}, \rho) := \bigcup_{\xi \in D_{a,b}} B(\xi, \rho). \tag{8}$$

We suppose that

$$r(K_2) < 1, \quad r(Q) < 1, \tag{9}$$

where

$$Q := \frac{3(b-a)}{10}\, K, \quad K = K_1 + K_2[I_n - K_2]^{-1} K_1. \tag{10}$$

On the base of function $f : [a,b] \times D \times D^1 \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\Big[\max_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y)\Big] \tag{11}$$

and suppose that the $\rho-$neighbourhood in (8) such that

$$\rho \geq \frac{b-a}{2}\, \delta_{[a,b],D,D^1}(f). \tag{12}$$

Investigation of solutions of parameterized problem (6) and (7) is connected with the properties of the following special sequence of functions well posed on the interval $t \in [a,b]$

$$x_0(t,z,\eta) := z + \frac{t-a}{b-a}\,[\eta - z] = \Big[1 - \frac{t-a}{b-a}\Big] z + \frac{t-a}{b-a}\,\eta, \quad t \in [a,b], \tag{13}$$

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\Big(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\, ds$$

$$-\frac{t-a}{b-a}\int_a^b f\Big(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\, ds + \frac{t-a}{b-a}\,[\eta - z], \quad t \in [a,b], \quad m = 0,1,2,\dots, \tag{14}$$

Theorem 1. *Let assumptions* (3)–(5) *and* (9) *hold. Then, for all fixed* $(z,\eta) \in D_a \times D_b$*:*

1. *The functions of the sequence* (14) *are continuously differentiable functions on the interval* $t \in [a,b]$*, have values in the domain* $D = D_\rho$ *and satisfy the two-point separated boundary conditions* (7)*.*

2. *The sequence of functions* (14) *in* $t \in [a,b]$ *converges uniformly as* $m \to \infty$ *to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{15}$$

satisfying the two-point separated boundary conditions (7)*.*

3. *The limit function* $x_\infty(t,z,\eta)$ *is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_a^t f\Big(s, x(s), \frac{dx(s)}{ds}\Big)\, ds - \frac{t-a}{b-a}\int_a^b f\Big(s, x(s), \frac{dx(s)}{ds}\Big)\, ds + \frac{t-a}{b-a}\,[\eta - z], \tag{16}$$

i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt} = f\Big(t, x, \frac{dx(t)}{dt}\Big) + \frac{1}{b-a}\,\Delta(z,\eta), \quad x(a) = z, \tag{17}$$

where $\Delta(z,\eta) : D_a \times D_b \to \mathbb{R}^n$ *is a mapping given by formula*

$$\Delta(z,\eta) := [\eta - z] - \int\limits_a^b f\Big(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\Big)\,ds. \tag{18}$$

4. *The following error estimate holds:*

$$\big|x_\infty(t,z,\eta) - x_m(t,z,\eta)\big| \leqslant \frac{10}{9}\,\alpha_1(t,a,b-a)Q^m(1_n - Q)^{-1}\delta_{[a,b],D,D^1}(f), \tag{19}$$

for any $t \in [a,b]$ *and* $m \geq 0$, *where* $\delta_{[a,b],D,D^1}(f)$ *is given in* (11) *and*

$$\alpha_1(t,a,b-a) = 2(t-a)\Big(1 - \frac{t-a}{b-a}\Big), \quad \alpha_1(t,a,b-a) \leq \frac{b-a}{2}\,. \tag{20}$$

Theorem 2. *Under the assumption of Theorem 1, the limit function* $x_\infty(t,z,\eta) : [a,b] \times D_a \times D_b \to \mathbb{R}^n$ *defined by* (15) *is a continuously differentiable solution of the original BVP* (1), (2) *if and only if the pair of vectors* (z,η) *satisfies the system of* $2n$ *determining algebraic equations*

$$\begin{cases} \Delta(z,\eta) = \eta - z - \displaystyle\int\limits_a^b f\Big(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\Big)\,ds = 0, \\ g\Big(x_\infty(a,z,\eta), x_\infty(b,z,\eta), \displaystyle\int\limits_a^b h(s, x_\infty(s,z,\eta))\,ds\Big) - d = 0. \end{cases} \tag{21}$$

Note that similarly as in [2] the solvability of the determining system (21) on the base of (3)–(5) and (9) can be established by studying its m-th approximate versions:

$$\begin{cases} \Delta_m(z,\eta) = \eta - z - \displaystyle\int\limits_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\Big)\,ds = 0, \\ g\Big(x_m(a,z,\eta), x_m(b,z,\eta), \displaystyle\int\limits_a^b h(s, x_m(s,z,\eta))\,ds\Big) - d = 0, \end{cases} \tag{22}$$

where m is fixed.

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \dfrac{1}{2}\,x_2^2(t) - t\,\dfrac{dx_2(t)}{dt}\,x_1(t) + \dfrac{1}{32}\,t^3 - \dfrac{1}{32}\,t^2 + \dfrac{9}{40}\,t, \\ \dfrac{dx_2(t)}{dt} = \dfrac{1}{2}\,\dfrac{dx_1(t)}{dt}\,x_1(t) - t^2x_2(t) + \dfrac{15}{64}\,t^3 + \dfrac{1}{8}\,t + \dfrac{1}{4}\,, \end{cases} \quad t \in [a,b] = [0,1], \tag{23}$$

considered with non-linear two-point boundary conditions

$$\begin{gathered} x_1(0)x_2(1) + \Big[\int\limits_0^1 x_1(s)\,ds\Big]^2 = -\frac{311}{14400}\,, \\ x_1(1)x_2(0) - \int\limits_0^1 x_2(s)\,ds = -\frac{1}{8}\,. \end{gathered} \tag{24}$$

Introduce the vector of parameters $z = col(z_1, z_2)$, $\eta = col(\eta_1, \eta_2)$. Let us consider the following choice of the subsets D_a, D_b and D^1:

$$D_a = D_b = \big\{(x_1, x_2): \ -0.1 \le x_1 \le 0.2, \ -0.2 \le x_2 \le 0.3\big\}, \tag{25}$$
$$D^1 = \bigg\{\Big(\frac{dx_1}{dt}, \frac{dx_2}{dt}\Big): \ -0.1 \le \frac{dx_1}{dt} \le 0.3, \ -0.1 \le \frac{dx_2}{dt} \le 0.3\bigg\}.$$

In this case $D_{a,b} = D_a = D_b$. For $\rho = col(\rho_1, \rho_2)$ involved in (12), we choose the vector $\rho = col(0.4; 0.4)$. Then, in view of (25) the set (8) takes the form

$$D = D_\rho = \big\{(x_1, x_2): \ -0.5 \le x_1 \le 0.6, \ -0.6 \le x_2 \le 0.7\big\}. \tag{26}$$

A direct computations show that the conditions (3), (9), (10) hold with

$$K_1 = \begin{bmatrix} 0.3 & 0.3 \\ 0.15 & 1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0.3367346939 & 0.5102040816 \\ 0.1836734694 & 1.051020408 \end{bmatrix}$$

and, therefore,

$$Q = \begin{bmatrix} 0.1010204082 & 0.1530612245 \\ 0.05510204082 & 0.3153061224 \end{bmatrix}, \quad r(Q) = 0.349278 < 1.$$

Furthermore, in view of (11)

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\Big[\max_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x)\in[a,b]\times D\times D^1} f(t,x,y)\Big] = \begin{bmatrix} 0.31 \\ 0.7325 \end{bmatrix},$$
$$\rho = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \ge \frac{b-a}{2}\,\delta_{[a,b],D,D_1}(f) = \begin{bmatrix} 0.155 \\ 0.36625 \end{bmatrix}.$$

We thus see that all the conditions of Theorem 1 are fulfilled, and the sequence of functions (14) for this example is uniformly convergent.

Applying Maple 14, we carried out the calculations.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10}, \quad x_2^*(t) = \frac{t}{4} \tag{27}$$

is a continuously differentiable solution of the problem (1), (2). For a different number of approximations m, we obtain from (22) the following numerical values for the introduced parameters which are presented in Table 1:

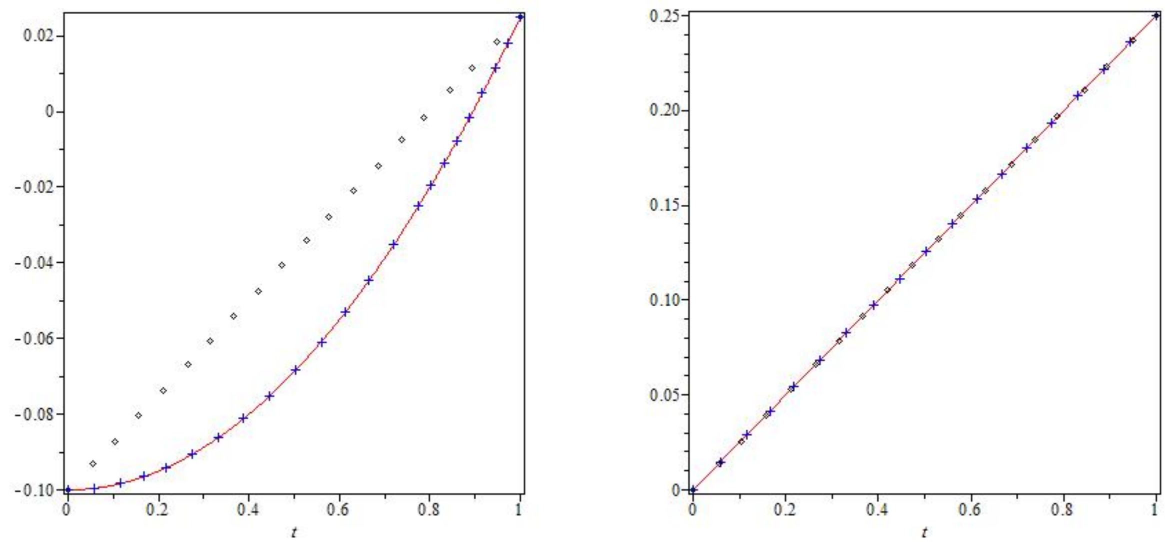
Table 1.

m	z_1	z_2	η_1	η_2
0	-0.089643967	-0.0002812586	0.03176891	0.25026338
1	-0.0994489263	0.00051937347	0.0255001973	0.2504687527
4	-0.0999998827	$7.744981 \cdot 10^{-8}$	0.02499999973	0.3535533902
6	-0.1000000004	$-2.263731 \cdot 10^{-10}$	0.0249999996	0.2499999996
Exact	-0.1	0	0.025	0.25

On the Figure 1 one can see the graphs of the exact solution (solid line) and its zero ($\Diamond$) and sixth approximation ($\times$) for the first and second coordinates.

The error of the sixth approximation ($m = 6$) for the first and second components:

$$\max_{t\in[0,1]} |x_1^*(t) - x_{61}(t)| \le 1 \cdot 10^{-9}, \quad \max_{t\in[0,1]} |x_2^*(t) - x_{62}(t)| \le 5 \cdot 10^{-9}.$$



a) first component

b) second component

Figure 1.

References

[1] A. Rontó, M. Rontó, J. Varha, A new approach to non-local boundary value problems for ordinary differential systems. *Appl. Math. Comput.* **250** (2015), 689–700.

[2] M. Rontó, Y. Varha, Constructive existence analysis of solutions of non-linear integral boundary value problems. *Miskolc Math. Notes* **15** (2014), no. 2, 725–742.