On Existence of Solutions with Prescribed Number of Zeros to High-Order Emden–Fowler Equations with Regular Nonlinearity and Variable Coefficient

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1 Introduction

The problem of existence of solutions with a countable number of zeros on a given domain to Emden–Fowler type equations is investigated. Consider the equation

$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ 0<m\leqslant p(t,\xi_1,\dots,\xi_n)\leqslant M<+\infty,\quad t\in\mathbb{R},\qquad(1.1)$$

where $n\in\mathbb{N}$, $n\geqslant 2$, $k\in\mathbb{R}$, $k>1$, the function $p(t,\xi_1,\dots,\xi_n)$ is continuous, and Lipschitz continuous in $\xi_1,\dots,\xi_n$.

We prove that equation (1.1) has solutions with a countable set of zeros on every finite interval $[a,b)$. The existence of solutions with a given finite number of zeros was considered in the previous papers, and results from them will be used to prove the main result. Namely, [3] is devoted to the case of the third- and the fourth-order Emden–Fowler type equations with constant p, [4,6] deal with the third-order equation with a variable coefficient, and [5,8] expand the previous results to the higher-order case. They based on the result obtained in [1,2]. Some results of the papers [3–6,8] can be summarized as

Theorem 1.1. *For any integer $S\geq 2$ and any finite interval $[a,b]\subset\mathbb{R}$ equation* (1.1) *has a solution $y(t)$ defined on the interval, $y(t)$ has exactly S zeros on the interval and $y(a)=0$, $y(b)=0$.*

Now, this theorem is expanded to the new case.

2 The main result

Theorem 2.1 ([7])**.** *For any finite interval $[a,b)\subset\mathbb{R}$ equation* (1.1) *has a solution $y(t)$ defined on the interval, $y(t)$ a countable set of zeros on the interval and $y(a)=0$.*

3 Sketch of the proof

The idea of the proof is similar to that of the proof of the main result from [8]. Suppose that $y(t)$ is a maximally extended solution to (1.1) with initial data $y(a)=0,y'(a)=y_1>0,\dots,y^{(n-1)}(a)=y_{n-1}>0$. In [1] it is proved that $y(t)$ has the countable number of zeroes. By t_N we denote a position of the N-th zero of $y(t)$ after the point a. In [8] it was proved that t_N is a continuous function on $(y_1,\dots,y_{n-1})$. Lower and upper estimates of the continuous function $t_N(y_1,\dots,y_{n-1})$ show that the N-th zero of the solution can be located at any point on the axis after a, hence solution with exactly N zeros can be defined on any $[a,b]$, if we choose appropriate initial data.

Proof of Theorem 2.1 has the same idea with some minor modifications. We know (see, for example, [1, Ch. 7]) that t_N tends to some finite limit t_* as $N \to +\infty$, but the solution itself is not defined at the point t_*. It appears that $t_*(y_1, \dots, y_{n-1})$ is also a continuous function of the variables $(y_1, \dots, y_{n-1})$ – like $t_N(y_1, \dots, y_{n-1})$. In addition, we obtain upper and lower estimates of t_* with the help of [1, p. 193, Lemmas 7.1, 7.2, 7.3] and Theorem 1.1.

We prove the continuity of $t_*(y_1, \dots, y_{n-1})$ using the continuity of every $t_N(y_1, \dots, y_{n-1})$ and lemmas [1, p. 193, Lemmas 7.1, 7.2, 7.3], since they give some estimates on the distance between t_N and t_{N+1} in comparison with the distance between t_N and t_{N-1}. The proposition of discontinuity of $t_*(y_1, \dots, y_{n-1})$ contradicts with those estimates.

4 Future plans

Papers [4, 5] demonstrate that Theorem 1.1 still holds true when $k \in (0, 1)$, so in future I hope to expand Theorem 2.1 on this case as well.

References

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