Nondecreasing Solutions of Singular Differential Equations

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1 Introduction

We investigate solutions of the initial value problem (IVP)

$$\big(p(t)\phi(u'(t))\big)' + p(t)f(\phi(u(t))) = 0, \ \ t \in (0,\infty), \tag{1.1}$$
$$u(0) = u_0, \quad u'(0) = 0, \ \ u_0 \in [L_0, 0), \tag{1.2}$$

where

$$\phi \in C^1(\mathbb{R}), \ \ \phi'(x) > 0 \ \text{ for } \ x \in (\mathbb{R} \setminus \{0\}), \tag{1.3}$$
$$\phi(\mathbb{R}) = \mathbb{R}, \ \ \phi(0) = 0, \tag{1.4}$$
$$L_0 < 0 < L, \ \ f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{1.5}$$
$$f \in \mathrm{Lip}[\phi(L_0), \phi(L)], \ \ xf(x) > 0 \ \text{ for } \ x \in \big((\phi(L_0), \phi(L)) \setminus \{0\}\big), \tag{1.6}$$
$$p \in C[0,\infty) \cap C^1(0,\infty), \ \ p'(t) > 0 \ \text{ for } \ t \in (0,\infty), \ \ p(0) = 0. \tag{1.7}$$

A function $u \in C^1[0,\infty)$ with $\phi(u') \in C^1(0,\infty)$ which satisfies equation (1.1) for every $t \in (0,\infty)$ is called a *solution* of equation (1.1). If moreover u satisfies the initial conditions (1.2), then u is called a *solution* of IVP (1.1), (1.2).

Equation (1.1) has the constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$.

Consider a solution u of IVP (1.1), (1.2) with $u_0 \in [L_0, 0)$ and denote

$$u_{sup} = \sup\big\{u(t) : \ t \in [0,\infty)\big\}.$$

- If $u_{sup} < L$, then u is called a *damped solution* of IVP (1.1), (1.2).
- If $u_{sup} = L$ and u is nondecreasing (i.e. $\lim\limits_{t\to\infty} u(t) = L$), then u is called a *homoclinic solution* of IVP (1.1), (1.2).
- The homoclinic solution is called a *regular homoclinic solution*, if $u(t) < L$ for $t \in [0,\infty)$ and a *singular homoclinic solution*, if there exists $t_0 > 0$ such that $u(t) = L$ for $t \in [t_0, \infty)$.
- If $u_{sup} > L$, then u is called an *escape solution* of IVP (1.1), (1.2).

In particular, we find additional conditions for p, ϕ and f which guarantee for some $u_0 \in [L_0, 0)$ the existence of a nondecreasing solution of IVP (1.1), (1.2) converging to L for $t \to \infty$. Note that if we extend the function p in equation (1.1) from the half-line onto $\mathbb{R}$ as an even function and assume that ϕ is odd, then any solution u of IVP (1.1), (1.2) with $\lim\limits_{t\to\infty} u(t) = L$ fulfils $\lim\limits_{t\to-\infty} u(t) = L$, hence u is *a homoclinic solution*. This is a motivation for our above definition. Due to condition (1.7) the function $1/p(t)$ may not be integrable on $[0,1]$ and consequently equation (1.1) has a time singularity at $t = 0$. Problems of this type arise in hydrodynamics [4] or in the nonlinear field theory [3], where homoclinic solutions play an important role in the study of behaviour of corresponding differential models.

Our first attempts in this subject have been made for the equation without ϕ-Laplacian

$$((t)u'(t))' + q(t)f(u(t)) = 0, \ \ t \in (0, \infty),$$

with $p \equiv q$ in [6–8] and for $p \not\equiv q$ in [1, 9].

2 Existence and asymptotic properties of solutions of IVP

Here we present an overview of results from [2] and [10] which we need to get a homoclinic solution of IVP (1.1), (1.2). Since values of any homoclinic solution belong to $[L_0, L]$, we can assume without loss of generality

$$f(x) = 0 \ \text{ for } \ x \le \phi(L_0), \ \ x \ge \phi(L). \tag{2.1}$$

Theorem 2.1 (Existence of solutions)**.** *Assume* (1.3)–(2.1)*. Then, for each starting value* $u_0 \in [L_0, 0)$*, there exists a solution of IVP* (1.1), (1.2)*.*

Theorem 2.2 (Damped solutions)**.** *Let* (1.3)–(2.1) *hold and let*

$$\exists\, \overline{B} \in (L_0, 0): \ F(\overline{B}) = F(L), \ \ \textit{where} \ \ F(x) = \int\limits_0^x f(\phi(s))\,\mathrm{d}s, \ \ x \in \mathbb{R}, \tag{2.2}$$

and

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{2.3}$$

Then every solution of IVP (1.1), (1.2) *with the starting value* $u_0 \in [\overline{B}, 0)$ *is damped.*

Assume in addition that

$$\lim_{x\to 0} |x|(\phi^{-1})'(x) < \infty, \tag{2.4}$$

and that u *is a damped solution of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, 0)$*. Then* u *is a unique solution of this IVP.*

Theorem 2.3 (Escape solutions)**.** *Let* (1.3)–(2.3) *hold. Then there exist infinitely many escape solutions of IVP* (1.1), (1.2) *with starting values in* $[L_0, \overline{B})$*.*

Assume in addition that (2.4) *hold and that* u *is an escape solutions of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, \overline{B})$*. Then* u *is a unique solution of this IVP.*

The next theorem describes asymptotic behaviour of damped, homoclinic and escape solutions starting at $u_0 \in (L_0, 0)$.

Theorem 2.4. *Let* (1.3)–(2.3) *hold and let* u *be a solution of IVP* (1.1), (1.2) *with the starting value* $u_0 \in (L_0, 0)$*. Then*

$$u(t) > L_0 \ \textit{ and } \ \exists\, \widetilde{c} > 0 \ \textit{ such that } \ |u'(t)| \le \widetilde{c} \ \textit{ for } \ t \in (0, \infty). \tag{2.5}$$

The constant $\widetilde{c}$ *depends on* L_0*,* L_1*,* ϕ *and* f *and does not depend on* p *and* u*.*

1. *Assume that* $u_{sup} < L$*, i.e.* u *is a damped solution.*

   - *Let* $\theta > 0$ *be the first zero of* u*. Then there exists* $\theta < a < b$ *such that*

$$u(a) \in (0, L), \quad u'(t) > 0 \ \textit{ on } (0, a), \quad u'(a) = 0, \quad u'(t) < 0 \ \textit{ on } (a, b). \tag{2.6}$$

- *Let $u<0$ on $[0,\infty)$. Then*

$$u'(t)>0 \ \ \textit{for} \ \ t\in(0,\infty), \quad \lim_{t\to\infty} u(t)=0, \quad \lim_{t\to\infty} u'(t)=0. \tag{2.7}$$

2. *Assume that $u_{sup}>L$, i.e. u is an escape solution. Then*

$$u'(t)>0 \ \ \textit{for} \ \ t\in(0,\infty). \tag{2.8}$$

3. *Assume that $u_{sup}=L$. Then there are two possibilities.*

- *$u(t)<L$ for $t\in[0,\infty)$ which yields*

$$u'(t)>0 \ \ \textit{for} \ \ t\in(0,\infty), \quad \lim_{t\to\infty} u(t)=L, \quad \lim_{t\to\infty} u'(t)=0, \tag{2.9}$$

and u is a regular homoclinic solution.

- *There exists $t_0>0$ such that $u(t_0)=L$, $u'(t_0)=0$ which implies*

$$u'(t)>0 \ \ \textit{for} \ \ t\in(0,t_0), \tag{2.10}$$

and there exists a singular homoclinic solution v, where $v=u$ on $[0,t_0]$ and $v=L$ on $[t_0,\infty)$.

Consider a solution $u\not\equiv L_0$ of IVP $(1.1),(1.2)$ with $u_0=L_0$. Since $L_0<0$, there exists $\varepsilon>0$ such that $u(t)<0$ for $t\in[0,\varepsilon]$, and by (2.1), $f(\phi(u(t)))\le 0$ for $t\in[0,\varepsilon]$. Integrating (1.1) over $[0,t]$ we get

$$p(t)\phi(u'(t))=-\int_0^t p(s)f(\phi(u(s)))\,\mathrm{d}s\ge 0, \quad t\in[0,\varepsilon].$$

Hence $u'(t)\ge 0$ and $u(t)$ is nondecreasing on $[0,\varepsilon]$. Consequently, since $u\not\equiv L_0$, there exists a maximal $a_0\ge 0$ such that

$$u(t)=L_0 \ \text{ on } [0,a_0] \text{ and } u \text{ is increasing in a right neighbouhood of } a_0. \tag{2.11}$$

Therefore all assertions of Theorem 2.4 are valid also for $u_0=L_0$ if we replace 0 by a_0.

3 Existence of homoclinic solutions

IVP $(1.1),(1.2)$ can be transformed on the equivalent integral equation

$$u(t)=u_0+\int_0^t \phi^{-1}\bigg(-\frac{1}{p(s)}\int_0^s p(\tau)f(\phi(u(\tau)))\,\mathrm{d}\tau\bigg)\,\mathrm{d}s, \quad t\in[0,\infty). \tag{3.1}$$

Assumption (1.3) implies that ϕ is locally Lipschitz continuous on $\mathbb{R}$, but if $\phi'(0)=0$, then

$$\lim_{x\to 0}(\phi^{-1})'(x)=\infty,$$

and so ϕ^{-1} does not fulfil the Lipschitz condition on intervals containing 0. If values of u are between L_0 and L, we see that

$$\lim_{s\to 0+}\frac{1}{p(s)}\int_0^s p(\tau)f(\phi(u(\tau)))\,\mathrm{d}\tau=0.$$

Therefore ϕ^{-1} in (3.1) is considered on an interval containing zero. Hence, in order to prove the uniqueness for IVP $(1.1),(1.2)$ if $\phi'(0)=0$, we need to use some new condition for ϕ^{-1} instead of the Lipschitz one. For such condition see (2.4). Then we get the main result published in [5] and contained in the next theorem.

Theorem 3.1 (Homoclinic solutions). *Let* (1.3)–(1.7) *and* (2.2)–(2.4) *hold. Further assume that*

$$\text{there exists a right neighbourhood of } \phi(L_0), \text{ where } f \text{ is decreasing.} \tag{3.2}$$

Then there exists $u_0^* \in [L_0, \overline{B})$ *such that a solution* u_h *of IVP* $(1.1),(1.2)$ *with* $u_0=u_0^*$ *is homoclinic.*

A typical model example of (1.1) is an equation with the α-Laplacian $\phi(x)=|x|^\alpha \operatorname{sgn} x$, $x\in\mathbb{R}$, where $\alpha\geq 1$. Then $\phi'(x)=\alpha|x|^{\alpha-1}$ and conditions (1.3) and (1.4) are fulfilled. If $\alpha>1$, then $\phi'(0)=0$, ϕ' is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,\infty)$. Further,

$$\phi^{-1}(x)=|x|^{\frac{1}{\alpha}}\operatorname{sgn} x, \quad (\phi^{-1})'(x)=\frac{1}{\alpha}\,|x|^{\frac{1}{\alpha}-1}, \quad \lim_{x\to 0}(\phi^{-1})'(x)=\infty,$$

which yields that ϕ^{-1} is not Lipschitz continuous at 0. Since

$$\lim_{x\to 0} x(\phi^{-1})'(x)=\frac{1}{\alpha}\lim_{x\to 0} x|x|^{\frac{1}{\alpha}-1}=0,$$

we see that the α-Laplacian $\phi(x)=|x|^\alpha\operatorname{sgn} x$ fulfils (2.4). If we take $p(t)=t^\beta$, $t\in[0,\infty)$, where $\beta>0$, then p fulfils (1.7). As an example of f satisfying conditions (1.5) and (1.6) we can choose

$$f(x)=x(x-\phi(L_0))(\phi(L)-x), \ \ x\in\mathbb{R}.$$

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