On a Weighted Problem for Functional Differential Equations with Decreasing Non-Linearity

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We study the weighted boundary value problem

$$u'(t) = (gu)(t), \ \ t \in (a,b], \tag{1}$$

$$\lim_{t\to a+} \varrho(t)u(t) \in \mathbb{R} \ \text{ exists}, \tag{2}$$

$$\int_a^b \varrho(t)|u'(t)|\,dt < +\infty, \tag{3}$$

where $-\infty < a < b < \infty$, $\varrho : (a,b] \to (0,+\infty)$ is a non-decreasing absolutely continuous function such that $\lim\limits_{t\to a+} \varrho(t) = 0$. We assume that $g : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$ is non-increasing in the sense that $(gu_1)(t) \le (gu_0)(t)$ for a.e. $t \in (a,b]$ for arbitrary pairs of functions $\{u_0,u_1\} \subset C((a,b],\mathbb{R})$ such that $u_1(t) \ge u_0(t)$, $t \in (a,b]$. In particular, the case of neutral type equations is excluded from consideration.

By a solution of equation (1), we mean a locally absolutely continuous function $u : (a,b] \to \mathbb{R}$ satisfying (1) almost everywhere on the interval $(a,b]$. In particular, solutions of (1) may be unbounded in a neighbourhood of the point a.

The formulation has been motivated, in particular, by a relation to boundary value problems with conditions at infinity, integral boundary conditions on unbounded intervals [1,3], and Kneser type solutions with possible blow-up [2,4].

The following notation is used.

$C((a,b],\mathbb{R})$ is the set of continuous functions $u : (a,b] \to \mathbb{R}$.

$L_1([a,b],\mathbb{R})$ is the set of Lebesgue integrable functions $u : [a,b] \to \mathbb{R}$.

$L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$ is the set of functions $u : (a,b] \to \mathbb{R}$ such that $u|_{[a_0,b]} \in L_1([a_0,b],\mathbb{R})$ for any $a_0 \in (a,b)$.

$\widetilde{C}([a,b],\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$\widetilde{C}_{\mathrm{loc}}((a,b],\mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a,b] \to \mathbb{R}$ (i.e., $u|_{[a_0,b]} \in \widetilde{C}([a_0,b],\mathbb{R})$ for any $a_0 \in (a,b)$).

$\widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ is the set of all $u \in \widetilde{C}_{\mathrm{loc}}((a,b],\mathbb{R})$ with $\varrho u' \in L_1((a,b],\mathbb{R})$ such that the limit $\lim\limits_{t\to a+} \varrho(t)u(t)$ exists and is finite.

Let ψ_0, ψ_1 be functions from $\widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ such that

$$(-1)^i(\psi_1^{(i)}(t) - \psi_0^{(i)}(t)) \ge 0, \ \ t \in (a,b], \ \ i = 0,1, \tag{4}$$

and

$$l_{\psi_0,\psi_1} := \inf\big\{\psi_1(t)-\psi_0(t):\ t\in(a,b]\big\}. \tag{5}$$

The value l_{ψ_0,ψ_1} is positive if the graphs of ψ_0 and ψ_1 do not touch each other. For any pair ψ_0, ψ_1 with the above properties, the set of functions u such that

$$\psi_0(t)+(1-\theta)l_{\psi_0,\psi_1}\le u(t)\le \psi_1(t)-\theta\, l_{\psi_0,\psi_1},\ \ t\in(a,b], \tag{6}$$

$$\psi_1'(t)\le u'(t)\le \psi_0'(t),\ \ t\in(a,b], \tag{7}$$

is non-empty for any $\theta\in[0,1]$. Introduce the set $S_\theta(\psi_0,\psi_1)$ by putting

$$S_\theta(\psi_0,\psi_1) := \big\{u\in \widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R}):\ (6)\text{ and }(7)\text{ hold}\big\} \tag{8}$$

for $\theta\in[0,1]$.

For any $\theta\in[0,1]$, the set $S_\theta(\psi_0,\psi_1)$ describes the area obtained by shifting the graphs of ψ_0 and ψ_1, respectively, upwards and downwards, in the ratio $1-\theta:\theta$, until they touch each other. Clearly, this happens at the points of the set

$$\big\{t\in(a,b]:\ \psi_1(t)-\psi_0(t)=l_{\psi_0,\psi_1}\big\}. \tag{9}$$

The typical situation is that where $(-1)^i\psi_i$, $i=0,1$, are non-decreasing and, hence, set (9) is a singleton consisting of the point b.

Theorem. *Let the mapping $g: C((a,b],\mathbb{R})\to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$ in* (1) *be non-increasing and, moreover,*

$$\varrho\, g\Big(\frac{\lambda}{\varrho}\Big)\in L_1((a,b],\mathbb{R}) \tag{10}$$

for any $\lambda\in\mathbb{R}$. Furthermore, let there exist certain functions ψ_0 and ψ_1 in $\widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ with properties (4) *such that*

$$(-1)^k\big(\psi_k'(t)-(g\psi_k)(t)\big)\ge 0,\ \ t\in(a,b],\ \ k=0,1. \tag{11}$$

Then for any $\theta\in[0,1]$ equation (1) *has a solution $u\in \widetilde{C}_{\mathrm{loc};\,\varrho}((a,b],\mathbb{R})$ such that $u\in S_\theta(\psi_0,\psi_1)$.*

Under the conditions assumed, one can guarantee the existence of solutions in the corresponding weighted space and specify certain bounds for u and u'. These bounds allow us to select solutions with different growth rates while we are still working in the same weighted space. Indeed, consider, e. g., the simple equation

$$u'(t)=\frac{\phi(u(1))}{t}-\frac{\psi(u(1))}{t^2}\,,\ \ t\in(0,1], \tag{12}$$

where $\phi(s)=2\pi^{-1}\operatorname{arccot} s-1/2$ and $\psi(s)=2\pi^{-1}\arctan s+1/2$ for all $s\in(-\infty,\infty)$. It is easy to see that any u satisfying (12) has the form

$$u_\lambda(t)=\lambda+\phi(\lambda)\ln t+\Big(\frac{1}{t}-1\Big)\,\psi(\lambda),\ \ t\in(0,1], \tag{13}$$

where $\lambda\in\mathbb{R}$, and since $|\phi(\lambda)|+|\psi(\lambda)|>0$, it follows that $u_\lambda(t)$ is unbounded as $t\to 0+$ for any λ. If $\lambda\ne -1$, then $\psi(\lambda)\ne 0$ and the growth of $|u_\lambda(t)|$ as $t\to 0+$ is of order $1/t$, whereas $u_{-1}(t)=-1+\ln t$ has only logarithmic growth. Note that the corresponding operator g for (12) is non-increasing.

For equation (12), conditions (4), (11) are satisfied, in particular, with

$$\psi_0(t)=0,\quad \psi_1(t)=\frac{1}{t}-1,$$

and, hence, the theorem claims that (12) has solutions u with the properties $0 \le u(t) \le -1 + 1/t$, $-1/t^2 \le u'(t) \le 0$, $u(1) = 0$, which indeed hold, e. g., for $u_0(t) = (\ln t + t^{-1} - 1)/2$ (see (13)). On the other hand, by choosing

$$\psi_0(t) = -1 + \mu \ln t, \quad \psi_1(t) = -1$$

with $\mu > 1$, we get the bounds $-1 + \mu \ln t \le u(t) \le -1$, $0 \le u'(t) \le \mu t^{-1}$, $u(1) = -1$ that fit only the solution $u_{-1}(t) = -1 + \ln t$ and do not cover u_λ with $\lambda \ne -1$. Note that (10) is satisfied in this case for $\rho(t) = t^\alpha$ with $\alpha > 1$.

If g is a linear operator of the form

$$(gu)(t) = -p(t)u(\tau(t)) + q(t), \ \ t \in (a, b],$$

where p and q are locally integrable, $p \ge 0$, and $\tau : (a, b] \to (a, b]$ is a measurable function, condition (10) reduces to the relations

$$\int\limits_a^b p(t)\, \frac{\varrho(t)}{\varrho(\tau(t))}\, dt < \infty, \quad \int\limits_a^b \varrho(t)|q(t)|\, dt < \infty, \tag{14}$$

which determine the corresponding class of equations for which the theorem can be applied. As an example, consider the linear equation with advanced argument

$$u'(t) = -\frac{u(t^\gamma)}{t} + q(t), \ \ t \in (0, 1], \tag{15}$$

where q is locally integrable and $\gamma \in (0, 1)$. The function $p(t) = 1/t$ satisfies (14) with $\varrho(t) = t^\alpha$, $t \in (0, 1]$, $\alpha > 1$. Then, for arbitrary $\mu > 0$, $\theta \in [0, 1]$, and q satisfying the estimate

$$|q(t)| \le \mu h(t), \ \ t \in (0, 1],$$

where $h(t) = t^{-2} - t^{-\gamma-1}$, $t \in (0, 1]$, the corresponding problem (15), (2), (3) has a solution u with the terminal value $u(1) = (1 - 2\theta)\mu$ such that

$$-\frac{\mu}{t} + 2(1 - \theta)\mu \le u(t) \le \frac{\mu}{t} - 2\theta\mu, \quad -\frac{\mu}{t^2} \le u'(t) \le \frac{\mu}{t^2}\,,$$

respectively, for all and almost all $t \in (0, 1]$. This follows from the theorem applied with $\psi_i(t) = (-\mu)^{i+1} t^{-1}$, $i = 0, 1$. Furthermore, if

$$-\mu h(t) \le \sigma q(t) \le \frac{\mu_0}{t}\,, \ \ t \in (0, 1],$$

for some $\sigma \in \{-1, 1\}$, $0 < \mu_0 \le \mu$, then for any $\theta \in [0, 1]$ there is a monotone solution with $u(1) = (\frac{1}{2}(\sigma + 1) - \theta)\mu + (\frac{1}{2}(\sigma - 1) + \theta)\mu_0$ such that

$$\mu \le \sigma u(t) + \Big(\sigma\theta + \frac{1 - \sigma}{2}\Big)(\mu - \mu_0) \le \frac{\mu}{t}\,, \ \ -\frac{\mu}{t^2} \le \sigma u'(t) \le 0.$$

In particular, for $q = -\sigma\mu h$, the problem in question admits the solution $u(t) = \sigma\mu t^{-1}$.

The conditions assumed do not exclude the possibility of existence of non-trivial solutions of homogeneous problems. For example, by taking $\psi_i(t) = (-1)^{i+1} \exp(2(t^{-2} - 1))$, $i = 0, 1$, we find that the equation

$$u'(t) = -\frac{4}{t^3}\, u\big(\sqrt{t}\,\big) + q(t), \ \ t \in (0, 1],$$

has a solution in the set $\widetilde{C}_{\mathrm{loc};\,\varrho}((0,1],\mathbb{R})$ for $\varrho(t)=\exp(-\alpha t^{-2})$, $\alpha>2$, if

$$|q(t)|\le \frac{4}{e^2t^3}\,e^{\frac{2}{t^2}}\big(1-e^{\frac{2(t-1)}{t^2}}\big),\ \ t\in(0,1].$$

One can verify by direct substitution that $u(t)=\frac{\lambda}{t^4}$ is a solution of the corresponding homogeneous problem for any λ.

The theorem ensures the existence of solutions lying between ψ_0 and ψ_1 with terminal values filling the corresponding interval. This does not exclude the possibility of existence of solutions which escape from the regions in question. For example, consider the functional differential equation

$$u'(t)=\frac{1}{t^2}\big(1-\exp(t)-u(\exp(-t))\big),\ \ t\in(0,1]. \tag{16}$$

Defining g according to the right-hand side of (16) and choosing the weight ϱ in the form $\varrho(t)=t^\alpha$, $\alpha>1$, we find that equation (16) satisfies conditions (10).

It is easy to verify that problem (16), (2), (3) with this ϱ has a one-parametric family of solutions

$$u(t)=-\frac{1}{t}-\lambda\ln t. \tag{17}$$

For $\psi_0(t)=-t^{-1}+2\ln t$, $\psi_1(t)=-t^{-1}-2\ln t$, the application of the theorem would result in the existence of solutions u such that

$$2\ln t\le u(t)+\frac{1}{t}\le -2\ln t,\quad -\frac{2}{t}\le u'(t)-\frac{1}{t^2}\le\frac{2}{t}\,,\quad u(1)=-1, \tag{18}$$

and such solutions are indeed obtained from (17) for $|\lambda|\le 2$. However, if $|\lambda|>2$, then solution (17) has the same terminal value -1 but does not satisfy conditions (18) any more.

In the cases where $\psi_0=c_0$ or $\psi_1=c_1$, where $c_0\le\psi_1(b)$ and $c_1\ge\psi_0(b)$, the solutions dealt with in the theorem are obviously monotone, and their terminal values fill, respectively, the intervals $[c_0,\psi_1(b)]$, $[\psi_0(b),c_1]$. With non-constant bounding functions, the solution, generally speaking, need not be monotone.

Under the conditions assumed, the set of solutions of the weighted problem in question possesses the least and the greatest elements.

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