The Generalized Jacobi–Poisson Theorem of Building First Integrals for Hamiltonian Systems

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1 Introduction

Consider the canonical Hamiltonian system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(t,q,p), \quad \frac{dp_i}{dt} = -\,\partial_{q_i} H(t,q,p), \ \ i = 1,\dots,n, \tag{1.1}$$

where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and $p = (p_1,\dots,p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, $t \in \mathbb{R}$, and the Hamiltonian $H : D \to \mathbb{R}$ is a twice continuously differentiable function on the domain $D = T \times G$, $T \subset \mathbb{R}$, $G \subset \mathbb{R}^{2n}$.

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(D)$ is the function

$$[u,v] : (t,q,p) \longrightarrow \sum_{i=1}^{n} \Big(\partial_{q_i} u(t,q,p)\, \partial_{p_i} v(t,q,p) - \partial_{p_i} u(t,q,p)\, \partial_{q_i} v(t,q,p) \Big) \ \text{ for all } \ (t,q,p) \in D.$$

A function $\mathrm{g} \in C^1(D')$ is called a *first integral on the domain* $D' \subset D$ of the Hamiltonian system (1.1) if $\mathfrak{G}\,\mathrm{g}(t,q,p) = 0$ for all $(t,q,p) \in D'$, where the linear differential operator

$$\mathfrak{G}(t,q,p) = \partial_t + \sum_{i=1}^{n} \big(\partial_{p_i} H(t,q,p)\, \partial_{q_i} - \partial_{q_i} H(t,q,p)\, \partial_{p_i} \big) \ \text{ for all } \ (t,q,p) \in D.$$

A smooth manifold $\mathrm{g}(t,q,p) = 0$ is said to be an *integral manifold* of the Hamiltonian system (1.1) if the derivative of the function $\mathrm{g} \in C^1(D')$ by virtue of the Hamiltonian system (1.1) is the identically zero on the manifold $\mathrm{g}(t,q,p) = 0$, i.e.,

$$\mathfrak{C}\,\mathrm{g}(t,q,p) = \Phi(t,q,p), \quad \Phi(t,q,p)\big|_{\mathrm{g}(t,q,p)=0} = 0 \ \text{ for all } \ (t,q,p) \in D'.$$

By $\mathrm{I}_{D'}$ $(\mathrm{M}_{D'})$ denote the set of all first integrals (integral manifolds) on the domain D' of the Hamiltonian system (1.1). The phrase "the function g is an integral manifold with function Φ on the domain D' of the Hamiltonian system (1.1)" is denoted by $(\mathrm{g}, \Phi) \in \mathrm{M}_{D'}$. For the current state of the theory of integrability see the monographs [2, 4, 5, 7–9] and the references therein.

Among the general methods of building first integrals of the Hamiltonian system (1.1), the Jacobi–Poisson method is of particular importance. It gives the possibility to find the additional (third) first integral of the Hamiltonian system (1.1) by two known first integrals of the Hamiltonian system (1.1). And thus, in certain cases, to build an integral basis of the Hamiltonian system (1.1). Due to this property, the Jacobi-Poisson method is included in almost all monographs and textbooks on analytical mechanics (see, for example, [6, pp. 298–306], [1, p. 216], [3, pp. 85–86]) and formulated as the following statement.

Theorem 1.1 (the Jacobi–Poisson theorem)**.** *Suppose twice continuously differentiable functions* $\mathrm{g}_1 : D' \to \mathbb{R}$ *and* $\mathrm{g}_2 : D' \to \mathbb{R}$ *are first integrals on the domain* D' *of the Hamiltonian system* (1.1)*. Then the Poisson bracket*

$$\mathrm{g}_{12} : (t,q,p) \longrightarrow \big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big] \ \textit{ for all } \ (t,q,p) \in D', \ \ D' \subset D, \tag{1.2}$$

of the functions g_1 *and* g_2 *is also a first integral of the Hamiltonian system* (1.1)*.*

In his *Lectures on Dynamics* [7, pp. 298–306], C. G. J. Jacobi referred to Poisson's theorem as "one of the most remarkable theorems of the whole of integral calculus. In the particular case when $H = T - U$, it is the fundamental theorem of analytical mechanics. . . . After I discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotton for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in a entirely different problem".

Of course, the Jacobi–Poisson theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the first integral obtained is simply a function of the original integrals. If neither of these two possibilities occurs, however, then the Poisson bracket is a further first integral of the Hamiltonian system (1.1).

The aim of this paper is to develop the Jacobi–Poisson method for integral manifolds of the Hamiltonian system (1.1).

2 Main results

Theorem 2.1. *Suppose* $(\mathrm{g}_k, \Phi_k) \in \mathrm{M}_{D'}$ *and* $\mathrm{g}_k \in C^2(D')$*,* $k = 1, 2$*. Then the Poisson bracket* $[\mathrm{g}_1, \mathrm{g}_2] \in \mathrm{I}_{D'}$ *if and only if the following identity holds*

$$\big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \,\big] = \big[\, \mathrm{g}_2(t,q,p), \Phi_1(t,q,p) \,\big] \ \textit{ for all } \ (t,q,p) \in D'. \tag{2.1}$$

Proof. Since $(\mathrm{g}_k, \Phi_k) \in \mathrm{M}_{D'}$, $k = 1, 2$, we have

$$\mathfrak{G}\, \mathrm{g}_k(t,q,p) = \Phi_k(t,q,p) \ \text{ for all } \ (t,q,p) \in D', \ \ k = 1, 2.$$

From these identities it follows that

$$\partial_t \mathrm{g}_k(t,q,p) = \Phi_k(t,q,p) - \big[\, \mathrm{g}_k(t,q,p), H(t,q,p) \,\big] \ \text{ for all } \ (t,q,p) \in D', \ \ k = 1, 2.$$

Using these identities and the properties of Poisson brackets (time derivative, bilinearity, anticommutativity, and Jacobi identity), we obtain the derivative of the function (1.2) by virtue of the Hamiltonian system (1.1)

$$\begin{aligned}
\mathfrak{G}\, \big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big] &= \partial_t \big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big] + \Big[\big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big], H(t,q,p) \Big] \\
&= \big[\, \partial_t \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big] + \big[\, \mathrm{g}_1(t,q,p), \partial_t \mathrm{g}_2(t,q,p) \,\big] + \Big[\big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big], H(t,q,p) \Big] \\
&= \Big[\Phi_1(t,q,p) - \big[\, \mathrm{g}_1(t,q,p), H(t,q,p) \,\big], \mathrm{g}_2(t,q,p) \Big] \\
&\quad + \Big[\mathrm{g}_1(t,q,p), \Phi_2(t,q,p) - \big[\, \mathrm{g}_2(t,q,p), H(t,q,p) \,\big] \Big] + \Big[\big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big], H(t,q,p) \Big] \\
&= \big[\, \Phi_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big] - \Big[\big[\, \mathrm{g}_1(t,q,p), H(t,q,p) \,\big], \mathrm{g}_2(t,q,p) \Big] + \big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \,\big] \\
&\quad - \Big[\mathrm{g}_1(t,q,p), \big[\, \mathrm{g}_2(t,q,p), H(t,q,p) \,\big] \Big] + \Big[\big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big], H(t,q,p) \Big]
\end{aligned}$$

$$= \big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \,\big] - \big[\, \mathrm{g}_2(t,q,p), \Phi_1(t,q,p) \,\big] + \Big(\Big[\big[\, H(t,q,p), \mathrm{g}_1(t,q,p) \,\big], \mathrm{g}_2(t,q,p) \Big]$$

$$+ \Big[\big[\, \mathrm{g}_2(t,q,p), H(t,q,p) \,\big], \mathrm{g}_1(t,q,p) \Big] + \Big[\big[\, \mathrm{g}_1(t,q,p), \mathrm{g}_2(t,q,p) \,\big], H(t,q,p) \Big] \Big)$$

$$= \big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \,\big] - \big[\, \mathrm{g}_2(t,q,p), \Phi_1(t,q,p) \,\big] \ \text{ for all } \ (t,q,p) \in D'.$$

Therefore the Poisson bracket (1.2) of the integral manifolds g_1 and g_2 of system (1.1) is a first integral of the Hamiltonian system (1.1) if and only if the identity (2.1) is true. □

Remark. If the function

$$\Phi : (t,q,p) \longrightarrow \big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \,\big] - \big[\, \mathrm{g}_2(t,q,p), \Phi_1(t,q,p) \,\big] \ \text{ for all } \ (t,q,p) \in D'$$

such that the following identity holds

$$\Phi(t,q,p)\big|_{[\mathrm{g}_1(t,q,p),\mathrm{g}_2(t,q,p)]=0} = 0 \ \text{ for all } \ (t,q,p) \in D',$$

then the Poisson bracket (1.2) is an integral manifold of the Hamiltonian system (1.1).

As a consequence of Theorem 2.1, we obtain

Corollary 2.1. *Let* $\mathrm{g}_1 \in \mathrm{I}_{D'}, (\mathrm{g}_2, \Phi_2) \in \mathrm{M}_{D'},\ \mathrm{g}_k \in C^2(D'),\ k = 1,2.$ *Then the Poisson bracket* $[\mathrm{g}_1, \mathrm{g}_2] \in \mathrm{I}_{D'}$ *if and only if the functions* g_1 *and* Φ_2 *are in involution, i.e.,*

$$\big[\, \mathrm{g}_1(t,q,p), \Phi_2(t,q,p) \big] = 0 \ \textit{ for all } \ (t,q,p) \in D'.$$

If $\mathrm{g}_1, \mathrm{g}_2 \in \mathrm{I}_{D'}$, then from Theorem 2.1 (or Corollary 2.1), we have the statement of the Jacobi–Poisson theorem (Theorem 1.1).

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