Stability Properties of Uniform Attractors for Parabolic Impulsive Systems

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An important problem in the theory of impulsive systems of differential equations [13] is a qualitative study of discontinuous (or impulsive) dynamical systems. In the case of an infinite-dimensional phase space, one of the most effective tools for studying the qualitative behavior of solutions is the theory of global attractors [4, 7]. The transfer of basic concepts and results of the theory of attractors to impulsive dynamical systems has a fundamental problem – the absence of continuous dependence of solutions on the initial data. Using the notion of a uniform attractor [4, 12], in [8], we were able to prove the existence of a minimal compact uniformly attracting set for a class of weakly nonlinear impulsive parabolic equations. Later in the works [5, 6, 9] this approach was extended to other classes of impulsive systems. It turned out that in the case when the trajectories of an impulsive dynamical system reach the impulsive set infinitely many times, the uniform attractor can have a non-empty intersection with the impulsive set and be neither invariant nor stable with respect to the impulsive semi-flow. The invariance of the non-impulsive part of a uniform attractor for different classes of impulsive systems was proved in [3, 5]. In [10], for the first time conditions for the impulsive semi-flow, which guarantee the stability of the non-impulsive part of the uniform attractors, were proposed. In this paper, we refine these conditions and apply them to study the stability of a uniform attractor of a weakly nonlinear two-dimensional impulsive-perturbed parabolic system.

Let us consider the impulsive dynamical system (further the impulsive DS) $G = G(V, M, I)$, which is defined on the normalized space X. It means that we consider the mapping $G : R_+ \times X \to X$, which is constructed from the continuous semigroup $V : R_+ \times X \to X$, the impulsive set $M \subset X$ and the impulsive map $I : M \to X$ using the following rule [11]: if for $x \in X$ for every $t > 0$ $V(t, x) \notin M$, then $G(t, x) = V(t, x)$; otherwise

$$G(t,x) = \begin{cases} V(t-t_n), & t \in [t_n, t_{n+1}), \\ x^+_{n+1}, & t = t_{n+1}, \end{cases} \tag{1}$$

where $t_0 = 0$, $t_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x^+_{n+1} = IV(s_n, x^+_n)$, $x^+_0 = x$, s_n are moments of impulsive perturbation, characterized by a condition $V(s_n, x^+_n) \in M$. Under conditions

$$\begin{gathered} M \ \text{ is closed}, \ \ M \cap IM = \varnothing, \\ \forall\, x \in M \ \ \exists\, \tau = \tau(x) > 0, \ \ \forall\, t \in (0, \tau) \ \ V(t,x) \notin M, \\ \forall\, x \in X \ \ t \to G(t,x) \ \text{ is defined on } \ [0, +\infty) \end{gathered} \tag{2}$$

formula (1) defines a semigroup $G : R_+ \times X \to X$ [2, 8].

Remark 1. From the condition (2) and the continuity of V follows [2, 5] that for every $x \in X$ either there is moments of time $s := s(x) > 0$ such that $\forall\, t \in (0, s)$ $V(t, x) \notin M$, $V(s, x) \in M$, or $\forall\, t > 0$ $V(t, x) \cap M = \varnothing$ (and in this case we set $s(x) = \infty$).

Definition 1 ([8])**.** A compact set $\Theta \subset X$ is called a uniform attractor of the impulsive DS G, if

1) Θ is uniformly attracting set, i.e.,

$$\forall\, B \in \beta(X) \quad \operatorname{dist}(G(t,B),\Theta) \longrightarrow 0, \ \ t \to \infty;$$

2) Θ is minimal closed set which satisfies 1).

Remark 2. A uniform attractor can be not invariant with respect to G. In that case the equality

$$\forall\, t \geq 0 \ \ \Theta = G(t,\Theta)$$

will not be fulfilled [8].

Theorem 1 ([5])**.** *Let impulsive* DS G *be dissipative, that is*

$$\exists\, B_0 \in \beta(X) \ \ \forall\, B \in \beta(X), \ \ \exists\, T = T(B) \ \ \forall\, t \geq T \ \ G(t,B) \subset B_0. \tag{3}$$

Then G has a uniform attractor Θ if and only if G is asymptotically compact, i.e. $\forall\,\{x_n\} \in \beta(X)$ $\forall\,\{t_n \nearrow \infty\}$ the sequence $\{G(t_n,x_n)\}$ is precompact. Herewith,

$$\Theta = \omega(B_0) := \bigcap_{\tau > 0} \overline{\bigcup_{t \geq \tau} G(t,B_0)}.$$

Definition 2 ([1])**.** The set $A \subset X$ is called a stable with respect to the semi-flow G, if

$$A = D^{+}(A) := \bigcup_{x \in A} \big\{ y \mid \ y = \lim G(t_n,x_n), \ x_n \to x, \ t_n \geq 0 \big\}. \tag{4}$$

In [10] it was shown that the uniform attractor of an impulsive DS may not satisfy the property (4), however, using additional assumptions about the nature of the behavior of the trajectories in the neighborhood of the impulsive set, we manage to obtain the following result which clarifies the statement of Theorem 1, 2 from [10].

Theorem 2. *Let impulsive* DS $G = (V,M,I)$ *satisfy conditions* (2), (3) *and have the uniform attractor Θ. Let impulsive mapping $I : M \to X$ and semi-group $V : R_+ \times X \to X$ be continuous and in addition, the conditions met:*

- *for an arbitrary sequence $x_n \to x \in \Theta \setminus M$*

$$\begin{cases} s(x) = \infty, & \textit{if } \ s(x_n) = \infty \ \textit{ for infinitely many } n, \\ s(x_n) \to s(x), & \textit{otherwise}; \end{cases}$$

- *for an arbitrary sequence $x_n \to x \in \Theta \cap M$*

$$\textit{either } \ s(x_n) = \infty \ \textit{ for infinitely many } \ n, \ \ \textit{or } \ s(x_n) \to 0.$$

Then the following equality is fulfilled:

$$\Theta = \overline{\Theta \setminus M}. \tag{5}$$

Moreover, Θ is invariant in the sense that

$$\forall\, t \geq 0 \ \ G(t,\Theta \setminus M) = \Theta \setminus M, \tag{6}$$

and stable in the sense that

$$D^{+}(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{7}$$

Let $\Omega \subset R^n$, $n \geq 1$ is a bounded domain. Using the unknown functions $u(t,x)$, $v(t,x)$ in $(0,+\infty) \times \Omega$ we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \varepsilon f_1(u,v), \\ \dfrac{\partial v}{\partial t} = a\Delta v + 2b\Delta u + \varepsilon f_2(u,v), \\ u\big|_{\partial\Omega} = v\big|_{\partial\Omega} = 0, \end{cases} \tag{8}$$

where $\varepsilon > 0$ is a small parameter,

$$a > 0, \quad |b| < a. \tag{9}$$

Nonlinear perturbation $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^1(R^2)$ satisfies the conditions:

$$\exists\, C > 0 \ \ \forall\, u, v \in R \quad |f_1(u,v)| + |f_2(u,v)| \leq C, \ \ Df(u,v) \geq -C, \tag{10}$$

which guarantee the single-valued global solvability of the problem (8) in a phase space $X = L^2(\Omega) \times L^2(\Omega)$ with the norm $\|z\|_X = \sqrt{\|u\|^2 + \|v\|^2}$, where here and further $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the scalar product in $L^2(\Omega)$.

Let $\{\lambda_i\}_{i=1}^{\infty} \subset (0,+\infty)$, $\{\psi_i\}_{i=1}^{\infty} \subset H_0^1(\Omega)$ be solutions of the spectral problem $\Delta\psi = -\lambda\psi$, $\psi \in H_0^1(\Omega)$.

For fixed $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\mu > 0$ the following impulsive problem is considered on the solutions of (8):

when the phase point $z(t)$ meets the impulsive set

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \ \middle|\ |(u,\psi_1)| \leq \gamma, \ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 \right\}, \tag{11}$$

it is instantly translated by the impulsive map $I : M \to M'$ to the new position $Iz \in M'$, where

$$M' = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \ \middle|\ |(u,\psi_1)| \leq \gamma, \ \alpha(u,\psi_1) + \beta(v,\psi_1) = 1 + \mu \right\}. \tag{12}$$

We will consider the following class of impulsive mappings:

$$\text{for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M, \quad I(z) = I_1 \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \psi_1 + \sum_{i=2}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M',$$

where $I_1 : R^2 \to R^2$ is specified continuous mapping.

In [9], it was proved that under the additional condition

$$2\beta\gamma \leq 1$$

the problem (8)–(12) for sufficiently small ε generates an impulsive DS G_ε which has a uniform attractor Θ_ε.

The main result of this paper is the following theorem.

Theorem 3. *Let $f_1 \equiv 0$. Then for sufficiently small $\varepsilon > 0$ the uniform attractor Θ_ε of the impulsive* DS G_ε, *generated by the problem* (8)–(12), *is invariant and stable in the sense* (5)–(7).

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