Conditions for Unique Solvability of the Two-Point Neumann Problem

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On a finite interval $[a,b]$, we consider the differential equation

$$u'' = f(t,u) \tag{1}$$

with the Neumann two-point boundary conditions

$$u'(a) = c_1, \quad u'(b) = c_2, \tag{2}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, while c_1 and c_2 are real constants.

A number of interesting and unimprovable in a certain sense results concerning the existence and uniqueness of a solution of problem (1), (2) are known (see, e.g., [1–8] and the references therein). Jointly with I. Kiguradze [9] we have proved a general theorem on the existence and uniqueness of a solution of that problem which is a nonlinear analogue of the first Fredholm theorem. Below we give this theorem and its corollaries containing unimprovable sufficient conditions, different from the above mentioned results, for the unique solvability of problem (1), (2).

We use the following notation.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}_- =]-\infty, 0]$;

$$[x]_- = \frac{|x| - x}{2};$$

$L([a,b])$ is the space of Lebesgue integrable on $[a,b]$ real functions.

Definition 1. Let $p_i \in L([a,b])$ $(i=1,2)$ and

$$p_1(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b]. \tag{3}$$

We say that the vector function (p_1, p_2) belongs to the set $\boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$ if for any measurable function $p : [a,b] \to \mathbb{R}$, satisfying the inequality

$$p_1(t) \le p(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b],$$

the homogeneous Neumann problem

$$u'' = p(t)u, \tag{4}$$

$$u'(a) = 0, \quad u'(b) = 0 \tag{5}$$

has only the trivial solution.

Theorem 1. *Let on the set $[a,b] \times \mathbb{R}$ the inequality*

$$p_1(t)|x-y| \le (f(t,x) - f(t,y)) \operatorname{sgn}(x-y) \le p_2(t)|x-y| \tag{6}$$

be satisfed, where $(p_1,p_2) \in \boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$. Then problem (1), (2) *has one and only one solution.*

Corollary 1. *Let on the set* $[a,b]\times\mathbb{R}$ *condition* (6) *hold, where* $p_i\in L([a,b])$ $(i=1,2)$ *are the functions satisfying inequalities* (3)*. Let, moreover,*

$$\int_a^b p_2(t)\,dt\le 0,\quad \operatorname{mes}\{[t\in[a,b]:\ p_2(t)<0\}>0, \tag{7}$$

and there exist a number $\lambda\ge 1$ *such that*

$$\int_a^b [p_1(t)]_-^{\lambda}\,dt\le \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda}. \tag{8}$$

Then problem (1), (2) *has one and only one solution.*

Corollary 2. *Let on the set* $[a,b]\times\mathbb{R}$ *inequality* (6) *hold, where* $p_1:[a,b]\to\mathbb{R}_-$ *and* $p_2:[a,b]\to\mathbb{R}$ *are integrable functions satisfying inequalities* (3) *and* (7)*. Let, moreover, there exist* $t_0\in\,]a,b[$ *such that the function* p_2 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$*, respectively, and*

$$\int_a^{t_0}\sqrt{|p_1(t)|}\,dt\le\frac{\pi}{2}\,,\quad \int_{t_0}^{b}\sqrt{|p_1(t)|}\,dt\le\frac{\pi}{2}\,,\quad \int_a^{b}\sqrt{|p_1(t)|}\,dt<\pi. \tag{9}$$

Then problem (1), (2) *has one and only one solution.*

The following two corollaries concern the linear differential equation

$$u''=p(t)u+q(t), \tag{10}$$

where p and $q\in L([a,b])$.

Corollary 3. *Let*

$$\int_a^b p(t)\,dt\le 0,\quad \operatorname{mes}\{t\in[a,b]:\ p(t)<0\}>0, \tag{11}$$

and let there exist a number $\lambda\ge 1$ *such that*

$$\int_a^b [p(t)]_-^{\lambda}\,dt\le \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda}. \tag{12}$$

Then problem (10), (2) *has one and only one solution.*

Corollary 4. *Let there exist a number* $t_0\in\,]a,b[$ *such that the function* p *along with* (11) *satisfies the conditions*

$$p_0(t)=\operatorname*{ess\,sup}\big\{[p(s)]_-:\ a<s<t\big\}<+\infty\quad\text{for } a<t<t_0, \tag{13}$$

$$p_0(t)=\operatorname*{ess\,sup}\big\{[p(s)]_-:\ t<s<b\big\}<+\infty\quad\text{for } t_0<t<b, \tag{14}$$

$$\int_a^{t_0}\sqrt{p_0(t)}\,dt\le\frac{\pi}{2}\,,\quad \int_{t_0}^{b}\sqrt{p_0(t)}\,dt\le\frac{\pi}{2}\,,\quad \int_a^{b}\sqrt{p_0(t)}\,dt<\pi. \tag{15}$$

Then problem (10), (2) *has one and only one solution.*

Remark 1. In the case, where instead of (11) the more hard condition

$$p(t) \le 0 \ \text{ for } \ a < t < b, \quad \operatorname{mes}\{t \in [a,b] : \ p(t) < 0\} > 0 \tag{16}$$

is satisfied, the results analogous to Corollary 3 previously were obtained in [4,5,8]. More precisely, in [8] it is required that along with (16) the inequalities

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}\,, \quad \operatorname{ess\,sup}\{|p(t)| : \ a \le t \le b\} < +\infty$$

be satisfied (see [8, Theorem 3]), while in [4] and [5] it is assumed, respectively, that

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}$$

(see [4, Corollary 1.2]), and

$$\int\limits_a^b |p(t)|^\lambda\,dt \le \frac{4(b-a)}{\pi^2}\,\Big(\frac{\pi}{b-a}\Big)^{2\lambda},$$

where $\lambda \equiv \mathit{const} \ge 1$ (see [5, Corollary 1.3]).

Example 1. Suppose

$$p(t) \equiv -\Big(\frac{\pi}{b-a}\Big)^2,$$

ε is arbitrarily small positive number, while λ is so large that

$$\Big(1 + \frac{\varepsilon}{\pi}\Big)^\lambda > \frac{\pi}{2}\,.$$

Then instead of (12) the inequality

$$\int\limits_a^b [p(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\,\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda} \tag{17}$$

is satisfied. On the other hand, the homogeneous problem (4), (5) has a nontrivial solution $u_0(t) = \cos\frac{\pi(t-a)}{b-a}$, and the nonhomogeneous problem (10), (2) has no solution if only

$$c_1 + c_2 + \int\limits_a^b u_0(t)q(t)\,dt \ne 0.$$

Consequently, condition (12) in Corollary 3 is unimprovable and it cannot be replaced by condition (17).

The above example shows also that condition (8) in Corollary 1 is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_a^b [p_1(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\,\Big(\frac{\pi+\varepsilon}{b-a}\Big)^{2\lambda},$$

where ε is a positive constant independent of λ.

Note that condition (8) in Corollary 1 is unimprovable also in the case where $\lambda = 1$, and it cannot be replaced by the condition

$$\int_a^b [p_1(t)]_- \, dt < \frac{4+\varepsilon}{b-a}$$

no matter how small $\varepsilon > 0$ would be (see [4, p. 357, Remark 1.1]).

Example 2. Suppose $t_0 \in]a,b[$ and

$$p(t) = \begin{cases} -\dfrac{\pi^2}{4(t_0-a)^2} & \text{for } \ a \le t \le t_0, \\ -\dfrac{\pi^2}{4(b-t_0)^2} & \text{for } \ t_0 < t \le b. \end{cases}$$

Then inequalities (13), (14) hold, and instead of (15) we have

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt = \frac{\pi}{2} \, , \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt = \frac{\pi}{2} \, .$$

On the other hand, the homogeneous problem (4), (5) has a nontrivial solution

$$u_0(t) = \begin{cases} (t_0-a)\cos\dfrac{\pi(t-a)}{2(t_0-a)} & \text{for } \ a \le t \le t_0, \\ (t_0-b)\cos\dfrac{\pi(b-t)}{2(b-t_0)} & \text{for } \ t_0 < t \le b, \end{cases}$$

while the nonhomogeneous problem (10), (2) has no solution if only

$$(t_0-a)c_1 + (b-t_0)c_2 + \int_a^b u_0(t)q(t) \, dt \neq 0.$$

Consequently, condition (15) in Corollary 4 is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt \le \frac{\pi}{2} \, , \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt \le \frac{\pi}{2} \, .$$

From the above said it is also clear that condition (9) in Corollary 2 is unimprovable and it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2} \, , \quad \int_{t_0}^b \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2} \, .$$

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