The Periodic Problem for the Second Order Integro-Differential Equations with Distributed Deviation

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On the interval $I=[0,\omega]$, consider the second order linear integro-differential equation

$$u''(t)=p_0(t)u(t)+\int\limits_0^{\omega} p(t,\,s)u(\tau(t,s))\,ds+q(t), \tag{0.1}$$

and the nonlinear functional differential equation

$$u''(t)=F(u)(t)+q(t), \tag{0.2}$$

with the periodic two-point boundary conditions

$$u^{(i-1)}(\omega)-u^{(i-1)}(0)=c_i \ \ (i=1,2), \tag{0.3}$$

where $c_1,c_2\in R$, $p_0,f,q\in L_\infty(I,R)$, $p\in L_\infty(I^2,R)$, $\tau:I^2\to I$ is a measurable function, and $F:C'(I,R)\to L_\infty(I,R)$ is a continuous operator.

By a solution of problem $(0.2),(0.3)$ we understand a function $u:I\to R$, which is absolutely continuous together with its first derivative, satisfies equation (0.2) almost everywhere on I and satisfies conditions (0.3).

Our work is motivated by some original results for the functional differential equations with argument deviation (see [1,2,4]), and the results of Nieto [5] and Kuo-Shou Chiu [3].

Here we establish theorems which in some sense complete and generalize the results of the works cited above as well as some other known results. We first describe some classes of unique solvability for linear problems $(0.1),(0.3)$, and then on the basis of these results, we prove the existence theorems for nonlinear problem $(0.2),(0.3)$. The conditions we obtain take into account the effect of argument deviation, and in some sense are optimal.

Throughout the paper we use the following notation.

$R=\,]-\infty,+\infty[\,$, $R_+=[0,+\infty[\,$.

$C(I;R)$ is the Banach space of continuous functions $u:I\to R$ with the norm $\|u\|_C=\max\{|u(t)|:\ t\in I\}$.

$C'(I;R)$ is the Banach space of functions $u:I\to R$ which are continuous together with their first derivatives with the norm $\|u\|_{C'}=\max\{|u(t)|+|u'(t)|:\ t\in I\}$.

$L(I;R)$ is the Banach space of Lebesgue integrable functions $p:I\to R$ with the norm $\|p\|_L=\int\limits_0^{\omega}|p(s)|\,ds$.

$L_\infty(I,R)$ is the space of essentially bounded measurable functions $p : I \to R$ with the norm $\|p\|_\infty = \operatorname{ess\,sup}\{|p(t)| : t \in I\}$.

$L_\infty(I^2,R)$ is the set of such functions $p : I^2 \to R$, that for any fixed $t \in I$, $p(t,\cdot\,) \in L(I,R)$, and $\int\limits_0^\omega |p(\,\cdot\,,s)|\,ds \in L_\infty(I,R)$.

Also for arbitrary $p_0, p_1 \in L_\infty(I,R)$, $p \in L_\infty(I^2,R)$, and measurable $\tau : I^2 \to I$ we will use the notation:

$$\ell_0(p_0,p)(t) = |p_0(t)| + \int\limits_0^\omega |p(t,s)|\,ds,$$

$$\ell_1(p,\tau) = \frac{2\pi}{\omega}\bigg(\int\limits_0^\omega \bigg(\int\limits_0^\omega |p(\xi,s)|\,|\tau(\xi,s)-\xi|\,ds\bigg)\,d\xi\bigg)^{1/2}.$$

Definition 0.1. Let $\sigma \in \{-1,1\}$, and $\tau : I \to I$ be the measurable function. We say that the vector-function $(h_0,h) : I \to R^2$, where $h_0 \in L_\infty(I,R_+)$ and $h \in L_\infty(I^2,R_+)$, belongs to the set P^σ_τ, if for an arbitrary vector-function $(p_0,p) : I \to R^2$ with such measurable components, that

$$0 \le \sigma p_0(t) \le h_0(t), \quad 0 \le \sigma p(t,s) \le h(t,s) \ \text{ for } \ t,s \in I,$$
$$p_0(t) + \int\limits_0^\omega p(t,s)\,ds \not\equiv 0, \tag{0.4}$$

the homogeneous problem

$$v''(t) = p_0(t)v(t) + \int\limits_0^\omega p(t,s)v(\tau(t,s))\,ds,$$
$$v^{(i-1)}(\omega) - v^{(i-1)}(0) = 0 \ \ (i=1,2),$$

has no nontrivial solution.

1 Linear problem

Proposition 1.1. *Let $\sigma \in \{-1,1\}$,*

$$h_0 \in L_\infty(I,R_+), \ \ h \in L_\infty(I^2,R_+), \quad h_0(t) + \int\limits_0^\omega h(t,s)\,ds \not\equiv 0,$$

and for almost all $t \in I$ the inequality

$$\frac{1-\sigma}{2}\,\ell_0(h_0,h)(t) + \ell_1(h,\tau)\ell_0^{1/2}(h_0,h)(t) < \frac{4\pi^2}{\omega^2}$$

holds. Then

$$(h_0,h) \in P^\sigma_\tau. \tag{1.1}$$

Theorem 1.1. *Let $\sigma \in \{-1,1\}$, $\sigma p_0 \in L_\infty(I,R_+)$, $\sigma p \in L_\infty(I^2,R_+)$, and condition* (0.4) *be fulfilled. Moreover, let for almost all $t \in I$ the inequality*

$$\frac{1-\sigma}{2}\,\ell_0(p_0,p)(t) + \ell_1(p,\tau)\ell_0^{1/2}(p_0,p)(t) < \frac{4\pi^2}{\omega^2} \tag{1.2}$$

hold. Then problem (0.1), (0.3) *is uniquely solvable.*

Let $p_0 \equiv 0$, $\tau(t,s) \equiv t - \nu(t,s)$, and $0 \le \nu(t,s) \le t$ for $t,s \in I$. Then equation (0.1) transforms to the next equation

$$u''(t) = \int\limits_0^\omega p(t,s)u(t-\nu(t,s))\,ds + q(t), \tag{1.3}$$

and from Theorem 1.1 it follows

Corollary 1.1. *Let conditions* $p \in L_\infty(I^2,R_+)$, $\int\limits_0^\omega p(t,s)\,ds \not\equiv 0$, *and for almost all* $t \in I$ *the inequality*

$$\int\limits_0^\omega \int\limits_0^\omega p(\xi,s)\nu(\xi,s)\,ds\,d\xi \int\limits_0^\omega p(t,s)\,ds < \frac{4\pi^2}{\omega^2}$$

hold. Then problem (1.3), (0.3) *is uniquely solvable.*

Corollary 1.2. *Let* $n \ge 3$, *and the function* $p_1 \in L_\infty(I,R_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$\int\limits_0^\omega \int\limits_0^t p_1(s)|\tau(s)-t|\,ds\,dt \int\limits_0^\omega p_1(s)\,ds \le \frac{4\pi^2[(n-3)!]^2}{\omega^{2(n-2)}}$$

holds. Then the problem

$$u^{(n)}(t) = p_1(t)u(\tau(t)) + q(t), \tag{1.4}$$

under the two-point boundary conditions

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = c_i, \ \ u^{(j-1)}(0) = c_j \ \ (i=1,2; \ \ j=3,\dots,n),$$

where $c_k \in R$ $(k=1,\dots,n)$, $p_1 \in L_\infty(I,R)$, *and* $\tau : I \to I$ *is a measurable function, is uniquely solvable.*

If $p_0 \equiv 0$ and $\tau(t,s) = \tau(t)$ for $t,s \in I$, then equation (0.1) transforms to the equation (1.4) with $n=2$, $p_1(t) = \int\limits_0^\omega p(t,s)\,ds$, and then from Theorem 1.1 it follows

Corollary 1.3. *Let* $p_1 \in L_\infty(I,R_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$p_1(t)\int\limits_0^\omega p_1(s)|\tau(s)-s|\,ds < \frac{4\pi^2}{\omega^2}$$

holds. Then problem (1.4), (0.3) *when* $n=2$ *is uniquely solvable.*

2 Nonlinear problem

Definition 2.1. We say that the operator F belongs to the Carathéodory's local class and write $F \in K(C', L_\infty)$, if $F : C'(I,R) \to L_\infty(I,R)$ is the continuous operator, and for an arbitrary $r > 0$,

$$\sup\big\{|F(x)(t)| : \ \|x\|_{C'} \le r, \ x \in C'(I,R)\big\} \in L_\infty(I,R_+).$$

Definition 2.2. Let $\sigma \in \{-1, 1\}$, inclusion (1.1) hold and the operators $V_0 : C'(I,R) \to L_\infty(I,R)$, $V : C'(I,R) \to L_\infty(I^2,R)$ be continuous. Then we say that $(V_0, V) \in E(h_0, h, P^\sigma_\tau)$, if for all $x \in C'(I,R)$ the conditions

$$0 \le \sigma V_0(x)(t) \le h_0(t), \ \ 0 \le \sigma V(x)(t,s) \le h(t,s) \ \text{ for } \ t,s \in I$$

hold, and

$$\inf \big\{ \|L(x,1)\|_L : \ x \in C'(I,R) \big\} > 0,$$

where

$$L(x,y)(t) = V_0(x)(t)y(t) + \int\limits_0^\omega V(x)(t,s)y(\tau(t,s))\,ds.$$

Also here it is assumed that the function sgn is defined by the equality

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } \ x \ge 0, \\ -1 & \text{if } \ x < 0. \end{cases}$$

Then the next theorem is true.

Theorem 2.1. *Let $\sigma \in \{-1,\, 1\}$, and*

$$(V_0 + \widetilde{V}_0, V) \in E(h_0, h, P^\sigma_\tau),$$

where the operators σV_0, $\sigma \widetilde{V}_0$ are nonnegative.

Moreover, let the constant $r_0 > 0$, the operator $F \in K(C', L_\infty)$, and the function $g_0 \in L(I, R_+)$, be such that the conditions

$$g_0(t) \le \sigma\big(F(x)(t) - L(x,x)(t)\big) \operatorname{sgn} x(t) \le \big|\widetilde{V}_0(x)(t)x(t)\big| + \eta\big(t, \|x\|_{C'}\big) \ \text{ for } \ t \in I, \ \ \|x\|_{C'} \ge r_0,$$

and

$$|c_2| \le \int\limits_0^\omega g_0(s)\,ds - \bigg| \int\limits_0^\omega q(s)\,ds \bigg|$$

hold, where the function $\eta : I \times R_+ \to R_+$ is summable in the first argument, nondecreasing in the second one, and admits the condition

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int\limits_0^\omega \eta(s,\rho)\,ds = 0.$$

Then problem (0.2), (0.3) *has at least one solution.*

References

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