On Unreachable Values of Boundary Functionals for Overdetermined Boundary Value Problems with Constraints

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1 Introduction

The classical formulation of the general linear boundary value problem (BVP) for linear ordinary differential system

$$(\mathcal{L}x)(t) \equiv \dot{x}(t) + A(t)x(t) = f(t), \quad t \in [0,T], \tag{1.1}$$

where $A(t)$ is a $n \times n$-matrix with elements summable on $[0,T]$, supposes that we are interested in the study of the question about the existence of solutions to (1.1) that satisfy the boundary conditions

$$\ell x = \beta \tag{1.2}$$

with linear bounded vector-functional $\ell = col(\ell_1, \dots, \ell_n)$ defined on the space of absolutely continuous functions $x : [0,T] \to R^n$ (see below more in detail). The key point in (1.1), (1.2) is that the number of linearly independent components ℓ_i in (1.2) equals the dimension of (1.1). In such a case, the unique solvability of BVP (1.1), (1.2) for $f = 0$, $\beta = 0$ implies the everywhere and unique solvability. If this is not the case, we have very specific situation with either the underdetermined BVP or the overdetermined BVP [11].

Linear BVP's for differential equations with ordinary derivatives, that lack the everywhere and unique solvability, are met with in various applications. Among these applications are some problems in Economic Dynamics [10, 12]. Results on the solvability and solutions representation for these BVP's are widely used as an instrument of investigating weakly nonlinear BVP's [6]. General results concerning linear BVP's for an abstract functional differential equation (AFDE) are given in [5]. As for linear overdetermined BVP's for AFDE in general, the principal results by L. F. Rakhmatullina are given in detail in [2, 3, 5].

In this paper, we consider the case that the number of linearly independent boundary conditions is greater than the dimension of the null-space of the corresponding homogeneous equation and study the BVP for FDE in an essentially different statement. Namely, the question we discuss is as follows: does there exist at least one free term f in the given linear FDE such that (1.2) holds for a fixed β, taking into account some given pointwise constraints with respect to $f(t)$ on $[0,T]$. Next we give a description for the set of unreachable β's, i.e. those for which f does not exist.

2 A class of boundary value problems

In this section, we consider a system of functional differential equations with aftereffect that, formally speaking, is a concrete realization of the AFDE, and, on the other hand, it covers many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference) [9, 12].

Let us introduce the functional spaces where operators and equations are considered. Fix a segment $[0,T] \subset R$. By $L_2^n = L_2^n[0,T]$ we denote the Hilbert space of square summable functions $v : [0,T] \to R^n$ endowed with the inner product $(u,v) = \int\limits_0^T u'(t)v(t)\,dt$ ($\cdot'$ is the symbol of transposition). The space $AC_2^n = AC_2^n[0,T]$ is the space of absolutely continuous functions $x : [0,T] \to R^n$ such that $\dot{x} \in L_2^n$ with the norm $\|x\|_{AC_2^n} = |x(0)| + \sqrt{(\dot{x},\dot{x})}$, where $|\cdot|$ stands for the norm of R^n.

Consider the functional differential equation

$$\mathcal{L}x \equiv \dot{x} - \mathcal{K}\dot{x} - A(\cdot)x(0) = f, \tag{2.1}$$

where the linear bounded operator $\mathcal{K} : L_2^n \to L_2^n$ is defined by

$$(\mathcal{K}z)(t) = \int\limits_0^t K(t,s)z(s)\,ds, \quad t \in [0,T],$$

the elements $k_{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and such that $|k_{ij}(t,s)| \le u(t)v(s)$, $i,j = 1,\dots,n$, $u,v \in L_2^1[0,T]$, $(n \times n)$-matrix A has elements that are square summable on $[0,T]$.

In what follows we will use some results from [1, 3, 8, 9] concerning (2.1). The homogeneous equation (2.1) ($f(t) = 0$, $t \in [0,T]$) has the fundamental $(n \times n)$-matrix $X(t)$:

$$X(t) = E_n + V(t),$$

where E_n is the identity $(n \times n)$-matrix, each column $v_i(t)$ of the $(n \times n)$-matrix $V(t)$ is a unique solution to the Cauchy problem

$$\dot{v}(t) = \int\limits_0^t K(t,s)\dot{v}(s)\,ds + a_i(t), \quad v(0) = 0, \quad t \in [0,T],$$

where $a_i(t)$ is the i-th column of A.

The solution to (2.1) with the initial condition $x(0) = 0$ has the representation

$$x(t) = (Cf)(t) = \int\limits_0^t C(t,s)f(s)\,ds,$$

where $C(t,s)$ is the Cauchy matrix [8] of the operator $\mathcal{L}$. This matrix can be defined (and constructed) as the solution to

$$\frac{\partial}{\partial t}\,C(t,s) = \int\limits_s^t K(t,\tau)\,\frac{\partial}{\partial \tau}\,C(\tau,s)\,d\tau + K(t,s), \quad 0 \le s \le t \le T,$$

under the condition $C(s,s) = E_n$. The properties of the Cauchy matrix used below are studied in detail in [9].

The matrix $C(t,s)$ is expressed in terms of the resolvent kernel $R(t,s)$ of the kernel $K(t,s)$. Namely,

$$C(t,s) = E_n + \int\limits_s^t R(\tau,s)\,d\tau.$$

The general solution to (2.1) has the form

$$x(t) = X(t)\alpha + \int\limits_0^t C(t,s)f(s)\,ds,$$

with an arbitrary $\alpha \in R^n$.

The general linear BVP is the system (2.1) supplemented by the linear boundary conditions

$$\ell x = \beta, \quad \beta \in R^N, \tag{2.2}$$

where $\ell : AC_2^n \to R^N$ is a linear bounded vector functional. Let us recall the representation of ℓ:

$$\ell x = \int\limits_0^T \Phi(s)\dot{x}(s)\,ds + \Psi x(0). \tag{2.3}$$

Here Ψ is a constant $(N \times n)$-matrix, Φ is $(N \times n)$-matrix with elements that are square summable on $[0,T]$. We assume that the components $\ell_i : AC_2^n \to R$, $i = 1,\dots,N$, of ℓ are linearly independent.

BVP (2.1), (2.2) is well-posed if $N = n$. In such a situation, the BVP is uniquely solvable for any $f \in L_2^n[0,T]$ and $\beta \in R^n$ if and only if the matrix

$$\ell X = (\ell X^1, \dots, \ell X^n),$$

where X^j is the j-th column of X, is nonsingular, i.e. $\det \ell X \neq 0$.

In the sequel we assume that $N > n$ and the system $\ell^i : AC_2^n \to R$, $i = 1,\dots,N$, can be splitted into two subsystems $\ell^1 : AC_2^n \to R^n$ and $\ell^2 : AC_2^n \to R^{N-n}$ such that the BVP

$$\mathcal{L}x = f, \quad \ell^1 x = \beta^1 \tag{2.4}$$

is uniquely solvable. Without loss of generality we will consider the case that ℓ^1 is defined by $\ell^1 x \equiv x(0)$, formed by the first n components of ℓ, and the elements of $\beta^1 = 0$ in (2.4) are the corresponding components of β. Thus ℓ^2 will stand for the final $(N-n)$ components of ℓ, and elements of $\beta^2 \in R^{N-n}$ are defined as the final $(N-n)$ components of β. Let us write ℓ_1 in the form

$$\ell^1 x = \int\limits_0^T \Phi_1(s)\dot{x}(s)\,ds + \Psi_1 x(0),$$

where $\Phi_1(s) = 0$ and $\Psi_1 = E_n$ are the corresponding rows of $\Phi(s)$ and Ψ, respectively, in (2.3). Similarly,

$$\ell_2 x = \int\limits_0^T \Phi_2(s)\dot{x}(s)\,ds + \Psi_2 x(0).$$

Put

$$\Theta_i(s) = \Phi_i(s) + \int\limits_s^T \Phi_i(\tau)C'_\tau(\tau,s)\,d\tau, \quad i = 1,2.$$

In the case that f is not constrained, it is shown in [11] that under the condition of nonsingularity of the matrix

$$W = \int\limits_0^T \Theta_2(s)\Theta_2'(s)\,ds \tag{2.5}$$

BVP (2.1), (2.2) is solvable for all $\beta^2 \in R^{N-n}$ if

$$f(t) = f_0(t) + \varphi(t),$$

where

$$f_0(t) = \Theta_2'(t)[W^{-1}\beta^2]$$

and $\varphi(\,\cdot\,) \in L_2^n$ is an arbitrary function that is orthogonal to each column of $\Theta_2'(\,\cdot\,)$:

$$\int\limits_0^T \Theta_2(s)\varphi(s)\,ds = 0.$$

Here we consider the case of the pointwise constraints

$$c_i \le f_i(t) \le d_i, \;\; t \in [0,T], \;\; c_i \le d_i, \;\; i = 1,\dots,n, \tag{2.6}$$

with respect to components $f_i(t)$ of the column $f(t) = col(f_1(t),\dots,f_n(t))$. Denote $\mathcal{V} = [c_1,d_1] \times \cdots \times [c_n,d_n]$.

In the sequel it is assumed that the elements of $\Phi_2(t)$ are piecewise continuous on $[0,T]$.

To formulate the main theorem, let us introduce some notation. For any $\lambda \in R^{N-n}$ and $t \in [0,T]$, we define $z(t,\lambda)$ by the equality

$$z(t,\lambda) = \max\big(\lambda'\Theta_2(t)v : \; v \in \mathcal{V}\big).$$

Define $v(t,\lambda)$ as the centroid of the collection of the unite mass points belonging to $\mathcal{V}$ and bringing the value $z(t,\lambda)$ to the functional $v \to \lambda' \cdot \Theta_2(t) \cdot v$.

Theorem. *Let a collection* $\{\lambda_i \in R^{N-n}, \; i = 1,\dots,m\}$ *be fixed, and a collection* $\{q_i \in R, \; i = 1,\dots,m\}$ *be such that the inequalities*

$$\lambda_i' \int\limits_0^T \Theta(t) \cdot v(t,\lambda_i)\,dt \le q_i, \;\; i = 1,\dots,m,$$

hold. Define $\mathcal{P}$ *as the set of all* $\rho \in R^{N-n}$ *such that the inequalities*

$$\lambda_i' \cdot \rho \le q_i, \;\; i = 1,\dots,m,$$

are fulfilled. Then all $\beta^2 \in R^{N-n}$ *outside the polyhedron* $\mathcal{P}$ *are unreachable for BVP* (2.1), (2.2) *under constraints* (2.6).

The proof of the theorem is based on [7, Theorem 7.1].

Example. Let us consider the system

$$\begin{aligned} \dot x_1(t) &= x_2(t-1) + f_1(t), \\ \dot x_2(t) &= -x_2(t) + f_2(t), \end{aligned} \quad t \in [0,3],$$

where $x_2(s) = 0$ if $s < 0$, with the initial conditions

$$x_1(0) = 0, \;\; x_2(0) = 0,$$

and additional conditions as follows:

$$x_1(3) - x_2(2) = \beta_1, \;\; x_2(3) + x_1(2) = \beta_2,$$

under the constraints

$$0 \le f_i(t) \le 2, \quad i = 1, 2.$$

Here we have

$$\begin{gathered}
C(t,s) = \begin{pmatrix} 1 & \int\limits_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix}, \\
\ell^2 x = col\big(x_1(3) - x_2(2), x_2(3) + x_1(2)\big), \\
\Theta_2(s) = \begin{pmatrix} C_{1,1}(3,s) - \chi_{[0,2]}(s)C_{2,1}(2,s) & C_{1,2}(3,s) - \chi_{[0,2]}(s)C_{2,2}(2,s) \\ C_{2,1}(3,s) + \chi_{[0,2]}(s)C_{1,1}(2,s) & C_{2,2}(3,s) + \chi_{[0,2]}(s)C_{1,2}(2,s) \end{pmatrix},
\end{gathered}$$

where $C_{j,k}(t,s)$, $j,k = 1,2$ are the components of $C(t,s)$. It should be noted that for W defined by (2.5) the inequality $\det W > 5$ holds.

By application of theorem for the case $\lambda_i = col(\sin(i\pi/4), \cos(i\pi/4))$, $i = 1,\dots,8$, we obtain that all points (β_1, β_2) outside the intersection of the quadrangle with corners $\{(-1.35, 1.10), (1.02, -1.30), (5.40, 7.90), \ (7.90, 5.50)\}$ and the quadrangle with corners $\{(-0.60, 0), (-0.60, 6.55), (7.05, 0), (7.05, 6.55)\}$ are unreachable for the problem under consideration.

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