On Adaptive Sequences to Evaluate Izobov Exponential Exponents

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Consider a linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A and with the Cauchy matrix X_A. Together with the system (1) consider a perturbed system

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \quad t \geq 0, \tag{2}$$

with piecewise continuous and bounded perturbation matrix Q. Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

One of the basic problem of Lyapunov exponents theory is to describe the influence of perturbations of coefficients from various classes on asymptotic properties of system (2). Usually these perturbations are considered as small in some sense. For example, the value $\Lambda(\mathfrak{M}, A) := \sup\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}\}$ is known as attainable bound of upward mobility of higher exponent of (2) with perturbations from $\mathfrak{M}$, see [4, p. 157], [8], [11, p. 39], [10, p. 46], [17]. The following classes are commonly used to calculate $\Lambda(\mathfrak{M}, A)$:

Infinitesimal perturbations [18]

$$Q(t) \to 0, \quad t \to +\infty, \tag{3}$$

exponentially small perturbations [9]

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)t), \quad C(Q) > 0, \quad \sigma(Q) > 0; \tag{4}$$

σ-perturbations [7]:

$$\|Q(t)\| \leq C(Q)\exp(-\sigma t), \quad C(Q) > 0, \quad \sigma > 0; \tag{5}$$

power perturbations

$$\|Q(t)\| \leq C(Q)t^{-\gamma}, \quad C(Q) > 0, \quad \gamma > 0; \tag{6}$$

generalized power perturbations [1, 2]

$$\|Q(t)\| \leq C(Q)\exp(-\sigma\theta(t)), \quad C(Q) > 0, \quad \sigma > 0, \tag{7}$$

$$\|Q(t)\| \leq C(Q)\exp(-\sigma(Q)\theta(t)), \quad C(Q) > 0, \quad \sigma(Q) > 0, \tag{8}$$

where θ is a positive function satisfying some additional conditions;

infinitesimal average [18] and integrable perturbations [3]

$$\lim_{t\to+\infty} \frac{1}{t}\int\limits_0^t \|Q(t)\|\,dt = 0, \qquad \int\limits_0^{+\infty} \|Q(t)\|\,dt < +\infty, \tag{9}$$

and their modifications with some positive weights φ and powers $p \ge 1$, see [4, p. 309], [5, 12, 13, 15, 16],

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \varphi(\tau)\|Q(\tau)\|^p \, d\tau = 0, \quad \int_0^{+\infty} \varphi(\tau)\|Q(t)\|^p \, dt < +\infty. \tag{10}$$

Sometimes [1–3, 12, 13, 15, 16] to calculate $\Lambda(\mathfrak{M})$ we can construct an algorithm analogous to a famous Izobov algorithm for σ-exponent [7]

$$\nabla_\sigma(A) = \overline{\lim_{k\to\infty}} \frac{\xi_k(\sigma)}{k}, \tag{11}$$
$$\xi_k(\sigma) = \max_{i\le k} \big\{ \ln \|X_A(k,i)\| + \xi_i(\sigma) - \sigma i \big\}, \quad \xi_0 = 0, \quad k \in \mathbb{N} \cup \{0\}.$$

For classes (5)–(7), (10), and the first of (9) we can write it in the general form

$$\Lambda(\mathfrak{M}, A) = \overline{\lim_{k\to\infty}} \frac{\ln \eta_k}{k}, \tag{12}$$
$$\eta_k = \max_{i\le k} \big\{ \|X_A(k,i)\|\beta(i)\eta_i \big\}, \quad \eta_0 = 1, \quad k \in \mathbb{N} \cup \{0\},$$

where $\beta(k)$, $\beta(0) > 0$ is some nonnegative function depending on $\mathfrak{M}$, e.g. $\beta(i) = e^{-\sigma i}$ for σ-perturbations. We shall consider β as a functional parameter of the algorithm.

The quantity η_k is always positive, because the maximum in (12) can not be reached at some $i \in \mathbb{N}$ if $\beta(i)$ is zero. We shall refer to this property of the algorithm (12) as adaptivity.

Alternatively, in some other cases [1, 2, 9, 18] we have formulas like the following Millionshcikov formula [4, p. 99], [8], [10, p. 48], [17]

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{k\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT - T)\|, \tag{13}$$

for the central exponent. One of such classes is the class of exponential perturbations, see formula (4). For exponential exponent $\nabla_0(A)$ corresponding to them [9] we have

$$\nabla_0(A) = \lim_{\theta\to1+0} \overline{\lim_{m\to\infty}} \frac{1}{\theta^m} \sum_{k=1}^{m} \ln \|X_A(\theta^k, \theta^{k-1})\|, \tag{14}$$

Also classes (3), (4), (8), and the second of (9) have the analogous expression for $\Lambda(\mathfrak{M}, A)$. The smallness classes $\mathfrak{M}$ for which $\Lambda(\mathfrak{M})$ has the representation of the form similar to (13), are called limit classes [1, 2].

One of the most important differences between representations (13) or (14) and algorithm (12) is that the sequence to calculate $\nabla_\sigma(A)$ is determined by system (1) itself, and $\nabla_0(A)$ or $\Omega(A)$ are calculated using strictly prescribed sequences. This rigidity does not allow us to construct analogues of formulas (13) and (14) for the perturbation classes with degenerations as it was done for algorithms of the type (12) in [14].

Let $\mathbb{T}$ be the set of all sequences $t_k \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, monotonically increasing to $+\infty$. For each $\tau \in \mathbb{T}$ put

$$\Omega(A, \tau) = \overline{\lim_{k\to\infty}} \frac{1}{t_{k+1}} \sum_{i=0}^{k} \ln \|X_A(t_{i+1}, t_i)\|.$$

We say that some family of sequences depending on a functional parameter β is adaptive if β is not zero at any element of each of these sequences.

We say that a one-parametric family $\mathbb{S}_\alpha$ of sequences is admissible for a class $\mathfrak{M}$ if for some α_0 the equality

$$\Lambda(\mathfrak{M}, A) = \lim_{\alpha\to\alpha_0} \sup_{\tau\in\mathbb{S}_\alpha} \Omega(A,\tau)$$

holds.

For any $\theta > 1$ $\quad\mathbb{T}_\theta$ by $\mathbb{T}$ let us denote the set of all sequences from $\mathbb{T}$ satisfying the condition $\varliminf\limits_{k\to+\infty} t_k^{-1} t_{k+1} \geq \theta$.

Lemma. *The equality*

$$\nabla_0(A) = \lim_{\theta\to 1+0} \sup_{\tau\in\mathbb{T}_\theta} \Omega(A,\tau)$$

holds.

Together with the property A established in [7] for the families of finite sequences implementing the σ-exponent $\nabla_\sigma(A)$, the above lemma allow us to give an algorithm for adaptive construction of sequences implementing the exponential exponent $\nabla_0(A)$. We can prove analogous lemmas for some other limit classes of perturbations.

Theorem. *For each of classes* (5)–(7)*, there exist a one-parametric family of admissible sequences.*

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