Global Components of Positive Bounded Variation Solutions of a One-Dimensional Capillarity Problem

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In this paper we study the topological structure of the set of positive bounded variation solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+u'^2}}\Big)' = \lambda a(x) f(u) \quad \text{in } (0,1),\\ u'(0)=0, \;\; u'(1)=0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^\infty(0,1)$ changes sign, $f \in C^1(\mathbb{R})$ satisfies $f(s)$, $s > 0$ for all $s \neq 0$ and $f'(0) = 1$. Problem (1) is a particular version of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) \quad \text{in } \Omega,\\ -\dfrac{\nabla u\cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This model plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [11, 19, 22–25, 30, 45, 46], capillarity phenomena for incompressible fluids [16, 20, 21, 27, 28], and reaction-diffusion processes where the flux features saturation at high regimes [12, 29, 44].

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with indefinite nonlinearities [1–3, 7, 8, 26, 33, 37], no results were available for the problem (2), even in the one-dimensional case (1), before [35, 36], where we began the analysis of the effects of spatial heterogeneities in the simplest prototype problem (1). Even if part of our discussion in this paper has been influenced by some results in the context of semilinear equations, it must be stressed that the specific structure of the mean curvature operator, $u \mapsto -\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$, makes the analysis in this paper much more delicate and sophisticated, as (1) may determine spatial patterns which exhibit sharp transitions between adjacent profiles, up to the formation of discontinuities [9, 10, 12, 17, 18, 29, 40, 42]. This special feature explains why the existence intervals of regular positive solutions of [14, 15, 39] are smaller than those given in the former references when dealing with bounded variation solutions. It is a well-agreed fact that the space of bounded variation functions is the most appropriate setting for discussing these topics. The precise notion of bounded variation solution of (1) used in this paper has been basically introduced in [5, 6] and it has been extensively used and discussed later (see, e.g., [35, 38, 40–43]).

Definition 1 (Bounded variation solution)**.** A bounded variation solution of problem (1) is a function $u \in BV(0,1)$ such that

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}}\,dx + \int_0^1 \frac{Du^s}{|Du^s|}\,D^s\phi = \int_0^1 \lambda a f(u)\phi\,dx \tag{3}$$

for all $\phi \in BV(0,1)$ such that $|D\phi^s|$ is absolutely continuous with respect to $|Du^s|$.

In Definition 1 the following notations are used for every $v \in BV(0,1)$ (we refer to, e.g., [4,13] for any required additional detail):

- $Dv = Dv^a dx + Dv^s$ is the Lebesgue–Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $Dv^a dx$, with density function Dv^a, and its singular part Dv^s, with respect to the Lebesgue measure dx in $\mathbb{R}$.
- $|Dv|$, $|Dv^a|$ and $|Dv^s|$ stand for the absolute variations of the measures Dv, Dv^a and Dv^s, respectively; thus, the Lebesgue–Nikodym decomposition of $|Dv|$ is given by
$$|Dv| = |Dv|^a\,dx + |Dv|^s = |Dv^a|\,dx + |Dv^s|.$$
- $\frac{Dv}{|Dv|}$ and $\frac{Dv^s}{|Dv^s|}$ denote the density functions of Dv and Dv^s, respectively, with respect to their absolute variations $|Dv|$ and $|Dv^s|$.

In [35], we discussed the existence and the multiplicity of positive bounded variation solutions of (1) under various representative configurations of the behavior at zero and at infinity of the function f. The solutions of [35] can be singular, for as they may exhibit jump discontinuities at the nodal points of the weight function a, while they are regular, at least of class C^1, on each open interval where the weight function a has a constant sign. Instead, in [36] we investigated the existence and the non-existence of positive regular solutions. Some of the most intriguing findings of [35,36] can be synthesized by saying that the solutions of (1) obtained in [35] are regular as long as they are small, in a sense to be precised later, whereas they develop singularities as they become sufficiently large. This is in complete agreement with the peculiar structure of the mean curvature operator, which combines the regularizing features of the 2-laplacian, when ∇u is sufficiently small, with the severe sharpening effects of the 1-laplacian, when ∇u becomes larger.

A natural question arising at the light of these novelties is the problem of ascertaining whether or not these regular and singular solutions can be obtained, simultaneously, by establishing the existence of connected components of bounded variation solutions bifurcating from $(l,u)=(l,0)$, which stem regular from $(l,0)$ and develop singularities as their sizes increase; thus establishing the coexistence along the same component of both regular and singular solutions, as synoptically illustrated by the two bifurcation diagrams in Figure 1. Although this phenomenology has been already documented by the special example of [36, Section 8], by means of a rather sophisticated phase plane analysis, solving this problem in our general setting still was a challenge.

The main aim of this work is establishing the existence of two connected components, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^+$, of the closure of the set of positive bounded variation solutions of problem (1),

$$\mathcal{S}^> = \big\{(\lambda,u) \in [0,+\infty) \times BV(0,1):\ u>0 \text{ is a solution of (1)}\big\} \cup \big\{(0,0),(\lambda_0,0)\big\},$$

emanating from the line $\{(l,0):\ l \in \mathbb{R}\}$ of the trivial solutions, at the two principal eigenvalues $l=0$ and $l=l_0$ of the linearization of (1) at $u=0$,

$$\begin{cases} -u'' = \lambda a(x)u & \text{in } (0,1),\\ u'(0)=u'(1)=0. \end{cases} \tag{4}$$

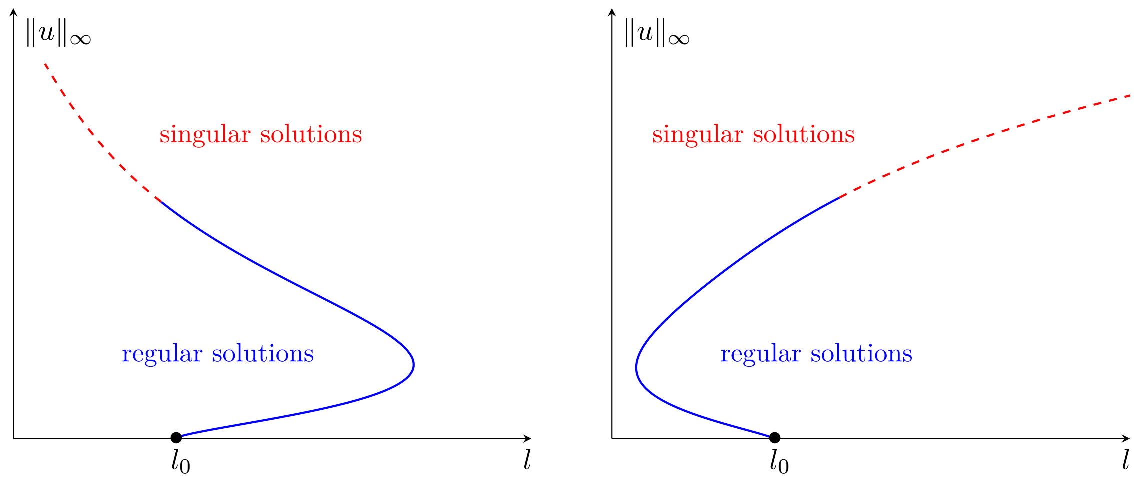

Figure 1. Global bifurcation diagrams emanating from the positive principal eigenvalue l_0, according to the nature of the potential $\int\limits_0^s f(t)\,dt$ of f: superlinear at infinity (on the left), or sublinear at infinity (on the right).

Precisely, our main global bifurcation theorem (see [34] for the proof) can be stated as follows.

Theorem 1. *Assume that $f \in C^1(\mathbb{R})$ satisfies $f(s)s > 0$ for all $s \neq 0$, $f'(0) = 1$, and, for some constants $\kappa > 0$ and $p > 2$, $|f'(s)| \leq \kappa\,(|s|^{p-2} + 1)$ for all $s \in \mathbb{R}$. Moreover, suppose that a satisfies $\int\limits_0^1 a(x)\,dx < 0$ and there is $z \in (0,1)$ such that $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$. Then there exist two subsets of $\mathcal{S}^>$, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ such that*

- *$\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ are maximal in $\mathcal{S}^>$ with respect to the inclusion, are connected with respect to the topology of the strict convergence in $BV(0,1)$[1], and are unbounded in $\mathbb{R} \times L^p(0,1)$;*
- *$(0,0) \in \mathcal{C}_0^>$ and $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}^>$;*
- *$\{(0,r) : r \in [0,+\infty)\} \subseteq \mathcal{C}_0^>$;*
- *if $(\lambda, u) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ and $u \neq 0$, then $\operatorname{ess\,inf} u > 0$;*
- *if $(\lambda, 0) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ for some $\lambda > 0$, then $\lambda = \lambda_0$;*
- *either $\mathcal{C}_0^> \cap \mathcal{C}_{\lambda_0}^> = \varnothing$, or $(\lambda_0, 0) \in \mathcal{C}_0^+$ and $(0,0) \in \mathcal{C}_{\lambda_0}^>$ and, in such case, $\mathcal{C}_0^> = \mathcal{C}_{\lambda_0}^>$;*
- *there exists a neighborhood U of $(0,0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_0^> \cap U$ consists of regular solutions of (1);*
- *there exists a neighborhood V of $(\lambda_0, 0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_{\lambda_0}^> \cap V$ consists of regular solutions of (1).*

Theorem 1 appears to be the first global bifurcation result for a quasilinear elliptic problem driven by the mean curvature operator in the setting of bounded variation functions. The absence in the existing literature of any previous result in this direction might be attributable to the fact that mean curvature problems are fraught with a number of serious technical difficulties which do not

[1] See [4, Definition 3.14]

arise when dealing with other non-degenerate quasilinear problems. As a consequence, our proof of Theorem 1 is extremely delicate, even though the problem (1) is one-dimensional. The main technical difficulties coming from the eventual lack of regularity of solutions of (1) as they grow, which does not allow us to work neither in spaces of differentiable functions, nor in Sobolev spaces. Instead, this lack of regularity forces us to work in the frame of the Lebesgue spaces L^p, where the cone of positive functions has empty interior and most of the global path-following techniques in bifurcation theory fail. Thus, to get most of the conclusions of Theorem 1, a number of highly non-trivial technical issues must be previously overcome. Among them count the reformulation of (1) as a suitable fixed point equation, the proof of the differentiability of the associated underlying operator, the search for the most appropriate global bifurcation setting, as well as solving the tricky problem of the preservation of the positivity of the solutions along both components, for as in the L^p context a positive solution, a priori, could be approximated by changing sign solutions. Naturally, none of these rather pathological situations cannot arise when dealing with classical regular problems, like those considered in [32].

For simplicity, here we have restricted ourselves to deal with the simplest situation when the function a possesses a single interior node z, and thus the positive solutions of (1) are monotone. As our proof relies, on a pivotal basis, on this special feature, getting a proof of this theorem in the general case when a has an intricate nodal behavior might be a real challenge plenty of technical difficulties. The validity of Theorem 1 in more general settings remains therefore an open problem.

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