On Instability of Millionshchikov Linear Systems with a Parameter

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We consider one-parameter family of linear differential systems

$$\dot{x} = A_\mu(t)x, \quad x \in \mathbb{R}^2, \quad t \geq 0 \tag{1_μ}$$

with the coefficient matrix $A_\mu(t) := d_k(\mu)\operatorname{diag}[1,-1]$, $2k-1 \leq t < 2k$, $A_\mu(t) := (\mu + b_k)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $2k-2 \leq t < 2k-1$, where $\mu, b_k \in \mathbb{R}$, $d_k(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}$.

In [4] we established the positivity of senior Lyapunov characteristic exponent of system (1_μ) for parameter values of positive Lebesgue measure, assumed that $d_k(\,\cdot\,)$ is independent on μ and the condition $d_k(\mu) \equiv d_k \geq d > 0$, $k \in \mathbb{N}$, holds. The proof of the result above substantially uses special complex matrices.

For all $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}$, let

$$b_{2^n} := b_{2^{n-1}} + \alpha_n, \quad b_{2^n+k} := b_k, \quad k = \overline{1, 2^n - 1}, \quad d_k(\mu) \equiv d_0(\mu) > 2^{20}, \quad k \in \mathbb{N}. \tag{2}$$

Systems of this type give rise to various one-parameter families with a wide range of asymptotic properties. For example, V. M. Millionshchikov used them in works [5, 6] (see, as well [3]) to prove an existence of irregular under Lyapunov linear differential systems with limit-periodic and quasi-periodic coefficients.

Method of these papers essentially use the estimations for eigenvalues and eigenvectors of system (1_μ) Cauchy matrix. Another way for investigation was initiated by the criterium due E. A. Barabanov of linear system regularity, that consist in the application of Cauchy matrix singular form (see the equality (5_n)).

In this paper we prove an existence of parameter value $\mu \in \mathbb{R}$ such that the corresponding system (1_μ) is unstable under condition (2) and if the function $d_0(\,\cdot\,)$ is continuous.

Let us denote the sequences $\{\psi_k(\mu)\}_{k=1}^{+\infty} \subset \mathbb{R}$ and $\{\eta_k(\mu)\}_{k=1}^{+\infty} \subset \mathbb{R}$ by the equalities $\psi_1(\mu) := \mu$, $\eta_1(\mu) = d_0(\mu)$, $\psi_{k+1} = \psi_k + \varphi_k/2$,

$$(\operatorname{ch}\eta_{k+1})\sin\varphi_k = \sin\xi_k, \quad k \in \mathbb{N}, \tag{3}$$

where $\xi_k := 2\psi_k + \zeta_k$, $\zeta_k := \sum\limits_{j=1}^{k} \alpha_j$, $\varphi_k \in (-2^{-1}\pi, 2^{-1}\pi]$ are defined by the formula

$$\operatorname{ctg}\varphi_k = (\operatorname{ch}2\eta_k)\operatorname{ctg}\xi_k. \tag{4}$$

Let $X_{A_\mu}(t,s)$, $t, s \geq 0$, is the Cauchy matrix for system (1_μ).

Lemma 1. *Foe all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$ under conditions* (2) *and* (3) *the next equalities hold*

$$X_{A_\mu}(2^n, 0) = U(\xi_n - \psi_n)\begin{pmatrix} \eta_n & 0 \\ 0 & \eta_n^{-1} \end{pmatrix} U(\psi_n), \tag{5_n}$$

$$\operatorname{sh}\eta_{k+1} = (\operatorname{sh}2\eta_k)\cos\xi_k. \tag{6}$$

Lemma 2. *For every continuous function $f(\,\cdot\,):[a,b]\to\mathbb{R}$, $a,b\in\mathbb{R}$, such that $f(a)\le c<d\le f(b)$, the closed interval $[p,q]\subset[a,b]$ exists with the property $f([p,q])=[c,d]$.*

Theorem. *For all $\alpha_n\in[-\pi/2,\pi/2]$, $n\in\mathbb{N}$, b_k and $d_k(\,\cdot\,)$, chosen accordinaly (2), the senior characteristic exponent of system (1_μ) is positive for some $\mu\in\mathbb{R}$, whereas the function $d_0(\,\cdot\,)$ is continuous.*

Proof. Let us denote

$$V_\varepsilon(\alpha):=\big\{\varkappa\in[-2^{-1}\pi,2^{-1}\pi]:\ |\sin(\varkappa-\alpha)|<\sin\varepsilon\big\}.$$

For every $k\in\mathbb{N}$ let

$$W_{k+1}:=[-2^{-1}\pi,2^{-1}\pi]\setminus\Big(\bigcup_{j=1}^{k}V_{2^{-j}-2^{-k-1}}(\zeta_j-2^{-1}\pi)\Big),\quad W_1:=(-\pi,\pi].$$

For all $j\in\{1,\dots,k\}$ a unic $\beta_{2j}(k),\beta_{2j+1}(k)\in(-2^{-1}\pi,2^{-1}\pi]$ exist such that

$$\sin(\beta_{2j+\delta}(k)-\zeta_j+2^{-1}\pi)=(-1)^\delta\sin(2^{-j}-2^{-k-1}),\ \ \delta\in\{0,1\}.$$

A substitution $j(\,\cdot\,):\{1,\dots,2k\}\to\{1,\dots,2k\}$ exist with the facility that the sequence $\{\beta_{j(i)}(k)\}_{i=1}^{2k}\subset(-2^{-1}\pi,2^{-1}\pi)$ do not decrease.

Let $\beta_{j(0)}:=-2^{-1}\pi$, $\beta_{j(2k+1)}:=2^{-1}\pi$.

The bound ∂W_{k+1} of the set W_{k+1} satisfies the inclusions

$$\partial W_{k+1}\subset\{-2^{-1}\pi,2^{-1}\pi\}\cup\Big(\bigcup_{j=1}^{k}\partial V_{2^{-j}-2^{-k-1}}(\zeta_j-2^{-1}\pi)\Big)\subset\{\beta_j(k)\}_{j=0}^{2k+1}. \tag{7}$$

We shall build the set $I_k\subset\{0,\dots,2k\}$ by the next way. Because of (7) for all $i\in\{0,\dots,2k\}$ or the relation $L_{i,k+1}:=[\beta_{j(i)},\beta_{j(i+1)}]\in W_{k+1}$ holds, in this case we set $I_k\ni i$, or, otherwise, the inclusion $L_{i,k+1}\in[-2^{-1}\pi,2^{-1}\pi]\setminus W_{k+1}$ is true. In the last case let $I_k\not\ni i$.

For every $i\in I_k$ let

$$b_i:=2^{-1}(\beta_{j(i)}+\beta_{j(i+1)})\in[-2^{-1}\pi,2^{-1}\pi],\quad c_i:=2^{-1}(\beta_{j(i+1)}-\beta_{j(i)})\in[-2^{-1}\pi,2^{-1}\pi].$$

Next equalities hold

$$L_{i,k+1}=\big\{\varphi\in[-2^{-1}\pi,2^{-1}\pi]:\ |\sin(\varphi-b_i)|\le\sin c_i\big\},\quad W_k=\bigcup_{i\in I_k}L_{i,k+1}.$$

If $k=0$, we set $I_0=1$, $L_{1,1}=[-2^{-1}\pi,2^{-1}\pi]$.

Assume the first that $\mu_{2j-1},\mu_{2j}\in\mathbb{R}$, $j\in I_{k-1}$, exist for some $k\in\mathbb{N}$ such that the equality holds

$$\sin\xi_k(M_{i,k})=\sin L_{i,k},\quad M_{i,k}:=[\mu_{2i-1},\mu_{2i}],\ \ i\in I_{k-1} \tag{8_k}$$

and, the second, that in the case $k>1$ we have the inclusion

$$M_k:=\bigcup_{j\in I_{k-1}}M_{j,k}\subset M_{k-1}. \tag{9_k}$$

Let us denote

$$s_k:=\sum_{j=1}^{k-1}2^{-j}j,\ \ s_1:=0.$$

Assume that the next inequality holds

$$\operatorname{sh}\ln\eta_k(\mu)\geq 2^{(9-s_k)2^k}. \tag{10_k}$$

Due to (8_k) for all $\mu\in M_k$ the inclusion $\xi_k(\mu)\in\mathbb{R}\setminus V_{2^{-k-1}}(\zeta_k-2^{-1}\pi))$ is true, that imply the inequalities

$$|\cos\xi_k(\mu)|\geq\sin 2^{-k-1}\geq 2^{-k-2}. \tag{11}$$

For all $\mu\in M_k$ the formulas (6), (10_k) and (11) give the estimation

$$\operatorname{sh}\ln\eta_{k+1}(\mu)\overset{(6)}{=}\operatorname{sh}\ln\eta_k^2\cos\xi_k(\mu)\overset{(11)}{\geq}2^{-k-2}\operatorname{sh}\ln\eta_k^2(\mu)\overset{(10_k)}{\geq}2^{(9-s_k)2^{k+1}-2k}\geq 2^{(9-s_{k+1})2^{k+1}}.$$

Hence we have the relation (10_{k+1}).

We set

$$S_k(\alpha):=\sum_{j=1}^k\alpha^j j.$$

For all $\alpha\in(-1,1)$ we obtain the equalities

$$S_{+\infty}(\alpha)=\Big(\sum_{j=1}^{+\infty}\alpha^j\Big)'_\alpha=\big((1-\alpha)^{-1}\big)'_\alpha=2(1-\alpha)^{-2}.$$

Since that the next relations hold

$$s_k\leq s_{+\infty}=\sum_{j=1}^{+\infty}2^{-j}j=S_{+\infty}(2^{-1})=8.$$

Hence, in view of (10_k), we have the estimate

$$\operatorname{sh}\ln\eta_k(\mu)\geq 2^{2^k}. \tag{12_k}$$

For all $i\in I_k$ the inclusion $V_{2^{-k-1}}(L_{i,k+1})\subset W_k$ is true. Since that, because of $L_{i,k+1}$ is the closed interval, there exists $j_i\in I_{k-1}$ such that the relation $V_{2^{-k-1}}(L_{i,k+1})\subset L_{j_i,k}$ holds.

Due to (4), (11) and (12_k), we have the estimates

$$\begin{aligned}&|\varphi_k(\mu)\leq 2|\sin\varphi_k(\mu)|\\&\qquad\leq 2|\operatorname{tg}\varphi_k(\mu)|\overset{(4)}{=}2(\operatorname{ch}2\eta_k(\mu))^{-1}\operatorname{tg}\xi_k(\mu)\leq 4e^{-2\eta_k(\mu)}|\cos\xi_k(\mu)|^{-1}\overset{(11),\,(12_k)}{\leq}2^{-k-1}.\end{aligned} \tag{13}$$

Hence the next inclusion holds

$$\psi_{k+1}(\mu_{2j-\delta})\overset{(12)}{\in}V_{2^{-k-1}}(\psi_k(\mu_{2j-\delta})),\quad \delta=\overline{0,1}. \tag{14}$$

Let us denote the function $f(\,\cdot\,):\mathbb{R}\to[-1,1]$ by the formula $f(\mu):=\sin\xi_{k+1}(\mu)$.

Because of (14) and due to (8_k), we have the inequality

$$|f(\mu_{2j-\delta})|\geq\sin(c_{j,k}-2^{-k-1})=:\varkappa. \tag{15}$$

Let us denote $s:=\operatorname{sgn}(f(\mu_{2j})-f(\mu_{2j-1}))$, $g(\mu):=sf(\mu)$.

The relation (15) implies the estimates

$$g(\mu_{2j-1,k})\leq-\varkappa<0<\varkappa\leq g(\mu_{2j,k}). \tag{16}$$

Because of continuity of the function $\eta_1(\,\cdot\,)$, $\varphi_{k+1}(\,\cdot\,)$ is also continuous, hence such is the function $g(\,\cdot\,)$. Since that and in view of (16) the function $g(\,\cdot\,)$ satisfies conditions of Lemma 2, in which one have denote $[a,b]:=[\mu_{2j-1,k},\mu_{2j,k}]$, $[c,d]:=[-\varkappa,\varkappa]$.

Hence, because of this lemma, there exists a closed interval $M_{i,k+1}:=[\mu_{2i-1,k+1},\mu_{2i,k+1}]\subset M_{j,k}$ such that $g(M_{i,k+1})=[-\varkappa,\varkappa]=\sin L_{i,k+1}$, that is (8_{k+1}) holds. Beside of that we have the inclusion (9_{k+1}).

Note that in the case $k=1$ the equalities $I_0=1$, $L_1=[-2^{-1}\pi,2^{-1}\pi]$ are true, since that, if denote $M_1:=M_{1,1}=[\mu_{1,1},\mu_{1,2}]:=[-2^{-1}\pi,2^{-1}\pi]$, we obtain the relation $\sin\xi_1(M_{1,1})=\sin([-\pi+a_1,\pi+a_1])=[-1,1]=\sin L_1$, that is, the equality (8_1) holds.

Due to (2), we have the inequalities

$$\operatorname{sh}\ln\eta_1(\mu)=2^{-1}(\eta_1(\mu)-\eta_1^{-1}(\mu))\overset{(2)}{\geq}2^{-1}(2^{20}-2^{-20})\geq 2^{18}=2^{(9-s_1)2^1},$$

that implies the estimates (10_1).

Under induction, we obtain the relations (8_n), (9_n) and (10_n) for every $1<n\in\mathbb{N}$.

Due to (8_k), the positivity of Lebesgue measure for the set W_k implies the inequality $M_k\neq\varnothing$. Hence, in view of (8_n), $n\in\mathbb{N}$, we have the existence of $\mu_{+\infty}\in M_{+\infty}:=\operatorname*{Lim}_{k\to+\infty}M_k$.

Because of (5_n) and (12_n), in view of the Lyapunov formula for the senior characteristic exponent of system (1_μ) [2], the next estimates hold

$$\lambda_{\max}(A_{\mu_{+\infty}})=\varlimsup_{t\to+\infty}t^{-1}\ln\|X_{A_{\mu_{+\infty}}}(t,0)\|\geq\varlimsup_{n\to+\infty}2^{-n}\ln\|X_{A_{\mu_{+\infty}}}(2^n,0)\|\overset{(5_n),(12_n)}{\geq}1.$$

They theorem is proved. □

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