The Asymptotic Behaviour of Solutions of Systems of Differential Equations Partially Solved Relatively to the Derivatives with Non-Square Matrices

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One of the methods of investigation of systems of differential equations which are not resolved relatively to the derivatives in the real-valued domain was suggested by R. Grabovskaya and J. Diblic [1]. It was developed in the complex domain in the articles by G. Samkova, N. Sharay, E. Michalenko, D. Limanska [2–6] and others. The current article is a continuation of the researching of systems of differential equations that are not resolved relatively to the derivatives in the complex domain.

Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z,Y,Y'), \tag{1}$$

where matrices $A, B : D_1 \to \mathbb{C}^{m\times p}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular on the condition that $z \to 0$, function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^m$, where domains $G_k \subset \mathbb{C}^p$, $0 \in G_k$, $k = 1, 2$, function $f(z,Y,Y')$ is analytic in $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

Let us study the system of ordinary differential equations (1) on the conditions that $m > p$ and $rang A(z) = p$ on condition that $z \in D_1$.

Without loss of the generality, let's assume that matrices $A(z)$, $B(z)$ and vector-function $f(z,Y,Y')$ take the forms

$$A(z) = \begin{pmatrix} A_1(z) \\ A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) \\ B_2(z) \end{pmatrix}, \quad f(z,Y,Y') = \begin{pmatrix} f_1(z,Y,Y') \\ f_2(z,Y,Y') \end{pmatrix},$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{(m-p)\times p}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{(m-p)\times p}$, $\det A_1(z) \neq 0$ on the condition that $z \in D_1$, $f_1 : D_1 \times G_1 \times G_2 \to C^p$, $f_2 : D_1 \times G_1 \times G_2 \to C^{m-p}$.

In this view the system (1) may be written as:

$$\begin{cases} Y' = A_1^{-1}(z)B_1(z)Y + A_1^{-1}(z)f_1(z,Y,Y'), & (2.1) \\ A_2(z)Y' = B_2(z)Y + f_2(z,Y,Y'), & (2.2) \end{cases} \tag{2}$$

where $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10}, $A_1^{-1}(z)f_1(z,Y,Y')$ is analytic vector-function in the domain $D_{10} \times G_{10} \times G_{20}$. Then vector-function $A_1^{-1}(z)f_1(z,Y,Y')$ has an isolated singularity in the point $(0,0,0)$. Thus, according to the theorem about an isolated singularity for a function of several complex variables, point $(0,0,0)$ is a removable singularity of the function $A_1^{-1}(z)f_1(z,Y,Y')$.

Let us complete definition of vector-function $A_1^{-1}(z)f_1(z,Y,Y')$ in the point $(0,0,0)$ thus it became analytic function in the domain $D_1 \times G_1 \times G_2$ and, without loss of the generality, let's assume that $A_1^{-1}(0)f_1(0,0,0) = 0$.

Let us consider two cases:

1. $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a removable singularity in the point $z = 0$;
2. $A_1^{-1}(z)B_1(z)$ is analytic matrix in the domain D_{10} and has a pole of order r in the point $z = 0$.

For the first case let us introduce the following notations

$$A_1^{-1}(z)B_1(z) = P^{(1)}(z), A_1^{-1}f_1(z, Y, Y') = F(z, Y, Y').$$

Then the system (2.1) may be written as

$$Y' = P^{(1)}(z)Y + F(z, Y, Y'), \tag{3}$$

where $P^{(1)} : D_1 \to \mathbb{C}^{p\times p}, P^{(1)}(z)$ is analytic matrix in the domain D_1, $F(z, Y, Y')$ is analytic vector-function in the domain $D_1 \times G_1 \times G_2$.

For the second case let us introduce the following notations

$$A_1^{-1}(z)B_1(z) = z^{-r}P^{(2)}(z), A_1^{-1}f_1(z, Y, Y') = F(z, Y, Y').$$

Then the system (2.1) may be written as

$$Y' = z^{-r}P^{(2)}(z)Y + F(z, Y, Y'), \tag{4}$$

where $P^{(2)} : D_1 \to \mathbb{C}^{p\times p}$, $P^{(2)}(z)$ is analytic matrix in the domain D_1.

We study the questions of the analytic solutions existence of the system (2) for both cases that satisfy the initial condition

$$Y(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}, \tag{5}$$

and additional condition

$$Y'(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}, \tag{6}$$

are considered.

The sufficient conditions of the existence of analytical solutions for the systems of differential equations (3) and (4), partially solved relatively to the derivatives, in the presence of a removable singularity or a pole z=0, were found. It was found an estimate for these solutions in the domain with the zero-point on a border.

The theorems on the existence of at least one analytic solution in the complex domain of the Cauchy problem (1)–(5) with the additional condition (6) are established for both cases. Moreover, the asymptotic behavior of these solutions in this domain is studied.

References

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