Asymptotic Representations of One Class Solutions of Second-Order Differential Equations

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Consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')[1+\psi(t,y,y')], \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ $(i=0,1)$ are continuous and regular varying as $y^{(i)} \to Y_i$ $(i=0,1)$ functions of orders σ_i $(i=0,1)$, Δ_{Y_i} $(i \in \{0,1\})$ is a one-side neighborhood of Y_i and $Y_i \in \{0;\pm\infty\}$ $(i \in \{0,1\})$, $\psi : [a,\omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is a continuous function such that the condition

$$\lim_{\substack{t\uparrow\omega \\ (y,z)\to(Y_0,Y_1) \\ (y,z)\in\Delta_{Y_0}\times\Delta_{Y_1}}} \psi(t,y,z) = 0$$

holds. We assume that the numbers μ_i $(i=0,1)$ given by the formula

$$\mu_i = \begin{cases} 1, & \text{if either } \ Y_i = +\infty, \ \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a right neighborhood of the point } 0, \\ -1, & \text{if either } \ Y_i = -\infty, \ \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a left neighborhood of the point } 0, \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \quad \text{and} \quad \mu_0\mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of Eq. (1) defined in the left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1). \tag{3}$$

We study Eq. (1) on class $P_\omega(Y_0,Y_1,\lambda_0)$-solutions, that is defined as follows.

Definition. A solution y of Eq. (1) on the interval $[t_0,\omega[\, \subset [a,\omega[$ is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition

$$\lim_{t\uparrow\omega} \frac{[y'(t)]^2}{y(t)y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. For $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$ in [1] such ratios

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0-1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0-1}\,,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t-\omega & \text{if } \ \omega < +\infty \end{cases}$$

are established.

By the definition of a regularly varying function [5, Chapter 1, Section 1.1, 9–10 of the Russian translation], each of the functions φ_i $(i \in \{0,1\})$ admits a representation of the form

$$\varphi_i(z) = |z|^{\sigma_i} L_i(z),$$

where $L_i : \Delta_{Y_i} \to]0,+\infty[$ is a continuous function slowly varying as $y \to Y_i$. Moreover, there exist continuously differentiable functions (see [5, Chapter 1, Section 1.1, 10–15 of the Russian translation]) $L_{ii} : \Delta_{Y_i} \to]0,+\infty[$ slowly varying as $y \to Y_i$ $(i = 0,1)$ and satisfying the conditions

$$\lim_{\substack{z \to Y_i \\ z \in \Delta_{Y_i}}} {}^{c} \frac{L_i(z)}{L_{ii}(z)} = 1, \quad \lim_{\substack{z \to Y_i \\ z \in \Delta_{Y_i}}} \frac{z L'_{ii}(z)}{L_{ii}(z)} = 0 \ \ (i = 0,1).$$

Asymptotic representations and conditions of the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions in case $\sigma_0 + \sigma_1 \neq 1$ are obtained in [4]. Here we study the behavior of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions in case $\sigma_0 + \sigma_1 = 1$ and $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$, when it becomes close in some sense to the linear, which is studied in detail in the monograph [3]. The theorem is a generalization of the result of work [2] for Eq. (1).

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \ \text{ for } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ for } \ Y_0 = +\infty \ \ (Y_0 = -\infty)$$

is respected and put

$$\begin{aligned} &\Delta_{Y_0}(b) = [b, Y_0[\ \text{ if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ &\Delta_{Y_0}(b) =]Y_0, b] \ \text{ if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0. \end{aligned}$$

Now we introduce auxiliary functions and notation as follows:

$$\Phi : \Delta_{Y_0}(b) \to \mathbb{R}, \quad \Phi(y) = \int\limits_B^y \frac{ds}{s L_0(s)}\,, \quad B = \begin{cases} b & \text{if } \displaystyle\int\limits_b^{Y_0} \frac{ds}{s L_0(s)} = \pm\infty, \\ Y_0 & \text{if } \displaystyle\int\limits_b^{Y_0} \frac{ds}{s L_0(s)} = const, \end{cases}$$

$$Z = \lim_{y \to Y_0} \Phi(y) = \begin{cases} 0 & \text{if } B = Y_0, \\ +\infty & \text{if } B = b, \ \mu_0\mu_1 > 0, \\ -\infty & \text{if } B = b, \ \mu_0\mu_1 < 0, \end{cases} \quad \mu_2 = \begin{cases} 1 & \text{if } B = b, \\ -1 & \text{if } B = Y_0, \end{cases}$$

$$I_0(t) = \int\limits_{A_0}^t p(\tau) |\pi_\omega(\tau)|^{-\sigma_1} L_1\big(\mu_1 |\pi_\omega(\tau)|^{\frac{1}{\lambda_0 - 1}}\big)\, d\tau, \quad I_1(t) = \int\limits_{A_1}^t p(\tau) |\pi_\omega(\tau)|^{\sigma_0} L_1\big(\mu_1 |\pi_\omega(\tau)|^{\frac{1}{\lambda_0 - 1}}\big)\, d\tau,$$

where the integration limits $A_i \in \{a; \omega\}$ $(i = 0,1)$ are chosen so as to ensure that the integrals I_i $(i = 0,1)$ tend either to zero or to $\pm\infty$ as $t \uparrow \omega$.

Theorem. *Let $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ and let the function $L_0(\Phi^{-1}(z))$ is regular varying of γ-th order as $z \to Z$, moreover, let the orders σ_i $(i = 0,1)$ of the functions φ_i $(i = 0,1)$ regularly varying as $y^{(i)} \to Y_i$ $(i = 0,1)$ satisfy the condition $\sigma_0 + \sigma_1 = 1$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the differential equation* (1)*, it is necessary and, if the condition*

$$(1 + \lambda_0)(1 + \lambda_0 + \lambda_0\gamma) \neq 0$$

is satisfied, sufficient that

$$\lim_{t\uparrow\omega}\frac{|\pi_\omega(t)|^{\sigma_0}p(t)L_1(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}})}{I_0(t)}=-\beta,\quad \lim_{t\uparrow\omega}\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)=Z,$$

$$\lim_{t\uparrow\omega}p(t)|\pi_\omega(t)|^{1+\sigma_0}L_1\big(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\big)L_0\Big(\Phi^{-1}\big(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)\big)\Big)=\frac{|\lambda_0|^{\sigma_0}}{|\lambda_0-1|^{1+\sigma_0}},$$

and the sign conditions

$$\mu_2\pi_\omega(t)I_1(t)>0,\quad \mu_0\mu_1\lambda_0(\lambda_0-1)\pi_\omega(t)>0\ \ for\ \ t\in\,]a,\omega[$$

hold. Moreover, each solution of this kind admits the asymptotic representations

$$\begin{aligned}\Phi(y(t))&=\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}&=\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}p(t)|\pi_\omega(t)|^{\sigma_0}\\ &\qquad\times L_1\big(\mu_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}\big)L_0\Big(\Phi^{-1}\big(\mu_0\mu_1|\lambda_0|^{\sigma_1}|\lambda_0-1|^{\sigma_0}I_1(t)\big)\Big)\ \ as\ \ t\uparrow\omega,\end{aligned}$$

and such solutions form a one-parameter family if

$$(\lambda_0-1)(1+\lambda_0+\gamma\lambda_0)I_1(t)<0\ \ for\ \ t\in\,]a,\omega[\,,$$

and two-parameter family if

$$(\lambda_0-1)(1+\lambda_0+\gamma\lambda_0)I_1(t)>0$$

and

$$(\lambda_0^2-1)\pi_\omega(t)>0\ \ for\ \ t\in\,]a,\omega[\,.$$

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