On Asymptotic Behavior of Solutions to Second-Order Differential Equations with General Power-Law Nonlinearities

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1 Introduction

Consider the second-order nonlinear differential equation

$$y'' = p(x, y, y')|y|^{k_0}|y'|^{k_1}\operatorname{sgn}(yy'), \quad k_0 > 0, \quad k_1 > 0, \quad k_0,\, k_1 \in \mathbb{R} \tag{1.1}$$

with positive continuous in x and Lipschitz continuous in u, v function $p(x, u, v)$ satisfying the inequalities

$$0 < m \le p(x, u, v) \le M < +\infty. \tag{1.2}$$

The results on the behavior of solutions depending on the nonlinearity exponents k_0, k_1 and qualitative properties of solutions was studied in [11].

The asymptoptic behavior of solutions to (1.1) in the case $k_1 = 0$ is described in [5,6]. In the case $p = p(x)$ asymptotic behavior of solutions to (1.1) is obtained by V. M. Evtukhov [7]. Using methods described in [1,2,4] by I. V. Astashova, the behavior of solutions to (1.1) near domain boundaries is considered with respect to the values k_0 and k_1.

The following definitions are used further.

Definition 1.1 ([4])**.** A solution $y : (a, b) \to \mathbb{R}$, $-\infty \le a < b \le +\infty$ to an ordinary differential equation is called a *μ-solution* if

(1) the equation has no other solutions equal to y on some subinterval (a, b) and not equal to y at some point in (a, b);

(2) the equation either has no solution equal to y on (a, b) and defined on another interval containing (a, b) or has at least two such solutions which differ from each other at points arbitrary close to the boundary of (a, b).

Definition 1.2 ([8])**.** A solution satisfying at some finite point x^* the conditions $\lim\limits_{x\to x^*} |y'(x)| = \infty$, $\lim\limits_{x\to x^*} |y(x)| < \infty$ is called a *black hole* solution.

Definition 1.3 ([9])**.** A μ-solution satisfying at finite point (its domain boundary) $\widetilde{x}$ the conditions $\lim\limits_{x\to\widetilde{x}} y'(x) = 0$ and $\lim\limits_{x\to\widetilde{x}} y(x) \ne 0$ is called a *white hole* solution.

Definition 1.4 ([10])**.** A solution to equation (1.1) is called a *Kneser solution at decreasing argument* on the interval $(-\infty; x_0)$ if $y(x) > 0$, $y'(x) > 0$ for any $x < x_0$.

Definition 1.5 ([10])**.** A solution to equation (1.1) is called a *negative Kneser solution* on the interval $(x_0; +\infty)$ if $y(x) < 0$, $y'(x) > 0$ for any $x > x_0$.

Definition 1.6 ([10])**.** A μ-solution $y(x)$ to equation (1.1) is called a *singular of the type II at a point* $a \in \mathbb{R}$ if $\lim\limits_{x\to a} y(x) = \lim\limits_{x\to a} y'(x) = 0$.

2 Main results

Lemma 2.1. *Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2)*. Then all μ-solutions to equation* (1.1) *are monotonous.*

Denote
$$\alpha = \frac{2-k_1}{k_0+k_1-1}\,, \quad C = \Big(\frac{|\alpha|^{1-k_1}|\alpha+1|}{p_0}\Big)^{\frac{1}{k_0+k_1-1}}.$$

Theorem 2.1. *Suppose $k_0+k_1<1$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2)*. Let there also exist the following limits of $p(x,u,v)$:*

(1) *p_+ as $x\to+\infty$, $u\to+\infty$, $v\to+\infty$;*

(2) *p_- as $x\to-\infty$, $u\to-\infty$, $v\to+\infty$.*

Denote $p_a=p(a,0,0)$ for any $a\in\mathbb{R}$. Then $\alpha<-1$ and all increasing μ-solutions to equation (1.1) *according to their asymptotic behavior can be divided into three types:*

1. *Increasing solutions defined on the whole axis with zero at some point x_0:*
$$y(x)=C(p_-)(x_0-x)^{-\alpha}(1+o(1)),\ \ x\to-\infty,$$
$$y(x)=C(p_+)(x-x_0)^{-\alpha}(1+o(1)),\ \ x\to+\infty.$$

2. *Positive singular solutions defined on semi-axis $(a,+\infty)$:*
$$y(x)=C(p_a)(x-a)^{-\alpha}(1+o(1)),\ \ x\to a+0,$$
$$y(x)=C(p_+)(x-a)^{-\alpha}(1+o(1)),\ \ x\to+\infty.$$

3. *Negative singular solutions defined on semi-axis $(-\infty,b)$:*
$$y(x)=C(p_-)(b-x)^{-\alpha}(1+o(1)),\ \ x\to-\infty,$$
$$y(x)=C(p_b)(b-x)^{-\alpha}(1+o(1)),\ \ x\to b-0.$$

Theorem 2.2. *Suppose $k_0+k_1>1$, $k_1<2$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (1.2)*. Let there also exist the following limits of $p(x,u,v)$:*

(1) *P^a as $x\to a-0$, $u\to+\infty$, $v\to+\infty$, for every $a\in\mathbb{R}$;*

(2) *P_a as $x\to a+0$, $u\to-\infty$, $v\to+\infty$, for every $a\in\mathbb{R}$;*

(3) *P_+ as $x\to+\infty$, $u\to0$, $v\to0$;*

(4) *P_- as $x\to-\infty$, $u\to0$, $v\to0$.*

Then $\alpha>0$ and all maximally extended increasing solutions to (1.1) *according to their asymptotic behavior can be divided into three types:*

1. *Increasing solutions with two vertical asymptotes $x=x_*$ and $x=x^*$, $x_*<x^*$:*
$$y=C(P^{x^*})(x^*-x)^{-\alpha}(1+o(1)),\ \ x\to x^*-0,$$
$$y=-C(P_{x_*})(x-x_*)^{-\alpha}(1+o(1)),\ \ x\to x_*+0.$$

2. *Kneser solution at decreasing argument defined on semi-axis* $(-\infty, x^*)$*:*

$$y = C(P_-)|x|^{-\alpha}(1+o(1)), \ \ x \to -\infty,$$
$$y = C(P^{x^*})(x^* - x)^{-\alpha}(1+o(1)), \ \ x \to x^* - 0.$$

3. *Negative Kneser solutions defined on semi-axis* $(x_*, +\infty)$*:*

$$y = -C(P_{x_*})(x - x_*)^{-\alpha}(1+o(1)), \ \ x \to x_* + 0,$$
$$y = -C(P_+)x^{-\alpha}(1+o(1)), \ \ x \to +\infty.$$

Theorem 2.3. *Suppose* $0 < k_1 < 1$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any maximally extended increasing solution* $y(x)$ *to* (1.1) *is a black hole solution defined on the interval* (x_*, x^*)*, and the limit* $\lim\limits_{x\to x^*-0} y(x) = y^*$ *satisfies the following inequalities:*

$$\Big(\frac{k_0+1}{M(k_1-2)}\Big)^{\frac{1}{k_0+1}} (y'(x_0))^{-\frac{k_1-2}{k_0+1}} \le |y^*| \le \Big(\frac{k_0+1}{m(k_1-2)}\Big)^{\frac{1}{k_0+1}} (y'(x_0))^{-\frac{k_1-2}{k_0+1}}.$$

The same inequalities hold for the limit $y_* = \lim\limits_{x\to x_*+0} y(x)$*.*

Theorem 2.4. *Suppose* $k_1 > 2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Let there also exist limits* p^+ *as* $x\to+\infty$*,* $u\to-\infty$*,* $v\to 0$ *and* p^- *as* $x\to-\infty$*,* $u\to-\infty$*,* $v\to0$*. Then* $-1<\alpha<0$ *and any increasing solution to* (1.1) *has a zero at some point* x_0 *and has the following asymptotic behavior:*

$$y(x) = -C(p^+)(x-x_0)^{-\alpha}(1+o(1)), \ \ x\to+\infty,$$
$$y(x) = C(p^-)(x_0-x)^{-\alpha}(1+o(1)), \ \ x\to-\infty.$$

Theorem 2.5. *Suppose* $k_0>0$*,* $1\le k_1<2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing solution* $y(x)$ *to equation* (1.1) *is defined on the whole axis, has a zero at some point* x_0 *and has two horizontal asymptotes* $y=y_+<0$ *at* $x\to+\infty$ *and* $y=y_->0$ *at* $x\to-\infty$*. Moreover,*

$$\frac{k_0+1}{M(2-k_1)}\,|y'(x_0)|^{2-k_1} \le |y_\pm|^{k_0+1} \le \frac{k_0+1}{m(2-k_1)}\,|y'(x_0)|^{2-k_1}.$$

Theorem 2.6. *Suppose* $k_0>0$*,* $0<k_1<1$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing* μ*-solution* $y(x)$ *to equation* (1.1) *is defined on a finite interval* (x_-, x_+)*, has a zero at some point* x_0 *and the limits* $y_+ = \lim\limits_{x\to x_+-0} y(x)$ *and* $y_- = \lim\limits_{x\to x_-+0}$ *satisfy the estimate from Theorem 2.5.*

Corollary 2.1. *Suppose* $k_0>0$*,* $0<k_1<2$*. Let the function* $p(x,u,v)$ *be continuous in* x*, Lipschitz continuous in* u*,* v *and satisfying inequalities* (1.2)*. Then any decreasing solution* $y(x)$ *to equation* (1.1) *is defined on the whole axis and the limits* $y_\pm = \lim\limits_{x\to\pm\infty} y(x)$ *satisfy the following inequalities:*

$$\Big(\frac{m}{M}\Big)^{\frac{1}{k_0+1}} \le \Big|\frac{y_+}{y_-}\Big| \le \Big(\frac{M}{m}\Big)^{\frac{1}{k_0+1}}.$$

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