# Asymptotic Behaviour of $P_\omega(Y_0,0)$-Solutions of Second-Order Nonlinear Differential Equations with Regularly and Rapidly Varying Nonlinearities

**N. P. Kolun**

*Military Academy, Odessa, Ukraine*
*E-mail:* nataliiakolun@ukr.net

Consider the differential equation

$$y''=\sum_{i=1}^{m}\alpha_i p_i(t)\varphi_i(y), \tag{1}$$

where $\alpha_i\in\{-1,1\}$ $(i=\overline{1,m})$, $p_i:[a,\omega[\,\to\,]0,+\infty[$ $(i=\overline{1,m})$ are continuous functions, $-\infty<a<\omega\le+\infty$; $\varphi_i:\Delta_{Y_0}\to\,]0,+\infty[$ $(i=\overline{1,m})$, where $\Delta_{Y_0}$ is a one-sided neighborhood of $Y_0$, $Y_0$ is equal either to zero or $\pm\infty$, are continuous functions for $i=\overline{1,l}$ and twice continuously differentiable for $i=\overline{l+1,m}$, and for each $i\in\{1,\dots,l\}$ for some $\sigma_i\in\mathbb{R}$

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi_i(\lambda y)}{\varphi_i(y)}=\lambda^{\sigma_i}\ \text{ for any }\ \lambda>0, \tag{2}$$

and for each $i\in\{l+1,\dots,m\}$ –

$$\varphi_i'(y)\neq0\ \text{ as }\ y\in\Delta_{Y_0},\quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\varphi_i(y)\in\{0,+\infty\},\quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)}=1. \tag{3}$$

It follows from the conditions (2) and (3) that $\varphi_i$ $(i=\overline{1,l})$ are regularly varying functions, as $y\to Y_0$, of orders $\sigma_i$ and $\varphi_i$ $(i=\overline{l+1,m})$ are rapidly varying functions, as $y\to Y_0$ (see [4, Introduction, pp. 2, 4]).

**Definition.** A solution $y$ of the differential equation (1) is called $P_\omega(Y_0,\lambda_0)$ – solution, where $-\infty\le\lambda_0\le+\infty$, if it is defined on some interval $[t_0,\omega[\,\subset[a,\omega[$ and satisfies the following conditions

$$\lim_{t\uparrow\omega}y(t)=Y_0,\quad \lim_{t\uparrow\omega}y'(t)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases}\quad \lim_{t\uparrow\omega}\frac{y'^2(t)}{y''(t)y(t)}=\lambda_0.$$

By its asymptotic properties, the class of $P_\omega(Y_0,\lambda_0)$ – solutions is split into 4 non-intersecting subsets that correspond to the next value of the parameter $\lambda_0$

$$\lambda_0\in\mathbb{R}\setminus\{0,1\},\quad \lambda_0=1,\quad \lambda_0=0,\quad \lambda_0=\pm\infty.$$

The existence conditions of $P_\omega(Y_0,\lambda_0)$ – solutions of the differential equation (1) and asymptotic representations, as $t\uparrow\omega$, of such solutions and their first-order derivatives, are established for each of these cases in the case where, for some $s\in\{1,\dots,m\}$

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))}=0\ \text{ for all }\ i\in\{1,\dots,m\}\setminus\{s\}, \tag{4}$$

i.e., where the right-hand side of Eq. (1) for each such solution $y$ is equivalent for $t \uparrow \omega$ to one term with regularly or rapidly varying nonlinearity (see [1–3]).

In this paper, we formulate the main results obtained for the case $\lambda_0 = 0$.

Let

$$\Delta_{Y_0} = \Delta_{Y_0}(b), \ \text{ where } \ \Delta_{Y_0}(b) = \begin{cases} [b, Y_0[\,, & \text{if } \ \Delta_{Y_0} \ \text{ is a left neighborhood of } \ Y_0, \\ ]Y_0, b], & \text{if } \ \Delta_{Y_0} \ \text{ is a right neighborhood of } \ Y_0, \end{cases}$$

and the number $b$ satisfy the inequalities

$$|b| < 1 \ \text{ as } \ Y_0 = 0 \ \text{ and } \ b > 1 \ \ (b < -1) \ \text{ as } \ Y_0 = +\infty \ \ (Y_0 = -\infty).$$

We set

$$\nu_0 = \operatorname{sign} b, \quad \nu_1 = \begin{cases} 1, & \text{if } \ \Delta_{Y_0}(b) = [b, Y_0[\,, \\ -1, & \text{if } \ \Delta_{Y_0}(b) = \,]Y_0, b], \end{cases} \quad \pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty, \end{cases}$$

$$J_{1s}(t) = \int\limits_{A_{1s}}^{t} p_s(\tau)\,d\tau, \quad J_{2s}(t) = \int\limits_{A_{2s}}^{t} J_{1s}(\tau)\,d\tau, \quad J_{3s}(t) = \int\limits_{A_{3s}}^{t} \pi_\omega(\tau) p_{0s}(\tau)\,d\tau,$$

$$H_s(y) = \int\limits_{B_s}^{y} \frac{du}{\varphi_s(u)}\,, \quad B_s = \begin{cases} b, & \text{if } \displaystyle\int\limits_{b}^{Y_0} \frac{dy}{\varphi_s(y)} = \pm\infty, \\ Y_0, & \text{if } \displaystyle\int\limits_{b}^{Y_0} \frac{dy}{\varphi_s(y)} = const, \end{cases} \qquad Z_s = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}(b)}} H_s(y),$$

$$J_{\varphi_s}(t) = \int\limits_{A_{\varphi_s}}^{t} p_{0s}(\tau) \varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(\tau))\big)\,d\tau, \quad E_s(t) = \alpha_s \pi_\omega^2(t) p_{0s}(t) \varphi_s'\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big),$$

$$G_s(t) = \frac{y\varphi_s'(y)}{\varphi_s(y)}\bigg|_{y = H_s^{-1}(-\alpha_s J_{3s}(t))}, \quad \Phi_s(t) = \frac{y\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)'}{\frac{\varphi_s'(y)}{\varphi_s(y)}}\Bigg|_{y = H_s^{-1}(-\alpha_s J_{3s}(t))},$$

$$\mu_s = \operatorname{sign} \varphi_s'(y), \quad \gamma_s = \lim_{t \uparrow \omega} \frac{E_s(t)\Phi_s(t)}{G_s(t)}\,, \quad \psi_s(t) = \int\limits_{t_0}^{t} \frac{|E_s(\tau)|^{\frac{1}{2}}}{\pi_\omega(\tau)}\,d\tau,$$

where $s \in \{1, \dots, m\}$, $p_{0s} : [a, \omega[\, \to ]0, +\infty[$ are continuous functions so that $p_{0s}(t) \sim p_s(t)$ as $t \uparrow \omega$, every limit of integration $A_{1s}$, $A_{2s}$, $A_{3s}$, $A_{\varphi_s}$ is equal to either $a$ or $\omega$ and is chosen so that the corresponding integral tends either to $\pm\infty$, or to zero with $t \uparrow \omega$, $t_0$ is some number of $[a, \omega[\,$.

**Theorem 1.** *Let $\sigma_s \neq 1$ for some $s \in \{1, \dots, l\}$ and there exist finite or equal to infinity limit*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_{1s}'(t)}{J_{1s}(t)}\,.$$

*For existence of $P_\omega(Y_0, 0)$ – solutions of equation (1), satisfied the limit relations (4), it is necessary that the inequalities*

$$\alpha_s \nu_0 (1 - \sigma_s) J_{2s}(t) > 0, \ \ \alpha_s \nu_1 \pi_\omega(t) < 0 \ \text{ as } \ t \in \,]a, \omega[ \tag{5}$$

*and conditions*

$$\alpha_s \lim_{t\uparrow\omega} J_{2s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{1s}(t)}{J_{1s}(t)} = -1, \quad \lim_{t\uparrow\omega} \frac{J^2_{1s}(t)}{p_s(t) J_{2s}(t)} = 0, \tag{6}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{1,\dots,l\} \setminus \{s\}, \tag{7}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)(1+\delta_i)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{l+1,\dots,m\}$$

*hold, where $\delta_i$ are arbitrary numbers of a one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations hold*

$$y(t) = H_s^{-1}(\alpha_s J_{2s}(t))[1+o(1)] \ \text{ at } \ t\uparrow\omega, \tag{8}$$

$$y'(t) = \frac{J_{1s}(t) H_s^{-1}(\alpha_s J_{2s}(t))}{(1-\sigma_s) J_{2s}(t)} [1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{9}$$

**Theorem 2.** *Let $\sigma_s \neq 1$ for some $s \in \{1,\dots,l\}$, conditions (5)–(7) hold and*

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)(1+u)))}{p_s(t)\varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \ \text{ for all } \ i \in \{l+1,\dots,m\}$$

*uniformly with respect to $u \in [-\delta, \delta]$ for any $0 < \delta < 1$. Then the differential equation (1) has $P_\omega(Y_0, 0)$ – solutions that admit the asymptotic representations (8) and (9). Moreover, if $\alpha_s \nu_0 (1-\sigma_s)\pi_\omega(t) < 0$ as $t \in ]a,\omega[$, there is a one-parameter family of such solutions in case $\omega = +\infty$ and two-parameter family in case $\omega < +\infty$.*

**Theorem 3.** *Let for some $s \in \{l+1,\dots,m\}$ the function $p_s$ might be representable in the form*

$$p_s(t) = p_{0s}(t)[1+r_s(t)], \ \text{ where } \ \lim_{t\uparrow\omega} r_s(t) = 0, \tag{10}$$

*$p_{0s} : [a,\omega[ \to ]0,+\infty[$ is a continuously differentiable function, $r_s : [a,\omega[ \to ]-1,+\infty[$ is a continuous function, and let the conditions*

$$\frac{\varphi_s(y)\varphi'_i(y)}{\varphi'_s(y)\varphi_i(y)} = O(1) \ \ (i = \overline{l+1,m}) \ \text{ for } \ y \to Y_0 \tag{11}$$

*hold. Then, for the existence of $P_\omega(Y_0, 0)$ – solutions of the differential equation (1) satisfying conditions (4), it is necessary that, there exist finite or equal to infinity limit*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{\varphi_s}(t)}{J_{\varphi_s}(t)},$$

*and the conditions*

$$\alpha_s \nu_1 \pi_\omega(t) < 0, \quad \alpha_s \mu_s J_{3s}(t) > 0 \ \text{ as } \ t \in ]a,\omega[, \tag{12}$$

$$-\alpha_s \lim_{t\uparrow\omega} J_{3s}(t) = Z_s, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J'_{\varphi_s}(t)}{J_{\varphi_s}(t)} = -1, \quad \lim_{t\uparrow\omega} \frac{\pi^2_\omega(t) p_{0s}(t) \varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))}{H_s^{-1}(-\alpha_s J_{3s}(t))} = 0, \tag{13}$$

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(-\alpha_s J_{3s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))} = 0 \ \text{ for all } \ i \in \{1,\dots,m\} \setminus \{s\} \tag{14}$$

*be satisfied. Moreover, each such solutions has the asymptotic representations*

$$y(t) = H_s^{-1}(-\alpha_s J_{3s}(t)) \Big[1 + \frac{o(1)}{G_s(t)}\Big] \ \text{ at } \ t\uparrow\omega, \tag{15}$$

$$y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)[1+o(1)] \ \text{ at } \ t\uparrow\omega. \tag{16}$$

**Theorem 4.** *Let for some $s \in \{l+1,\dots,m\}$ the conditions* (10), (11), (12)–(14) *be satisfied and*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_{3s}'(t)}{J_{3s}(t)} = \eta_s, \quad \text{where } \eta_s \in \mathbb{R}.$$

*Then:*

1) *if $\eta_s > 0$ or $\eta_s = 0$ and $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,0)$ – solutions with the asymptotic representations* (15) *and* (16)*;*

2) *if $\eta_s < 0$ or $\eta_s = 0$ and $\alpha_s\mu_s = -1$, there is a two-parameter family of $P_\omega(Y_0,0)$ – solutions which admit the asymptotic representations* (15), (16) *in case $\omega < +\infty$ and there is at least one such solution in case $\omega = +\infty$.*

**Theorem 5.** *Let for some $s \in \{l+1,\dots,m\}$ the function $p_s$ be representable in the form* (10)*, let conditions* (11), (12)–(14) *hold, and let the limits* (*which are finite or equal to $\pm\infty$*)

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_{\varphi_s}''(t)}{J_{\varphi_s}'(t)}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}(b)}} \frac{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)'}{\big(\frac{\varphi_s'(y)}{\varphi_s(y)}\big)^2}\cdot\sqrt{\Big|\frac{y\varphi_s'(y)}{\varphi_s(y)}\Big|}, \quad \gamma_s = \lim_{t\uparrow\omega} \frac{E_s(t)\Phi_s(t)}{G_s(t)}, \quad \lim_{t\uparrow\omega} \frac{\psi_s''(t)\psi_s(t)}{\psi_s'^2(t)}$$

*exist. Then:*

1) *if $\alpha_s\mu_s = 1$, the differential equation* (1) *has a one-parameter family of $P_\omega(Y_0,0)$ – solutions which admit the asymptotic representations* (15) *and* (16) *and are such that their derivatives satisfy the asymptotic relation*

$$y'(t) = -\alpha_s\pi_\omega(t)p_{0s}(t)\varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)\big[1 + |E_s(t)|^{-\frac12} o(1)\big] \ \text{ at } \ t\uparrow\omega;$$

2) *if $\alpha_s\mu_s = -1$ and*

$$\gamma_s \neq \frac13; \quad \lim_{t\uparrow\omega} \psi_s(t) r_s(t) = 0, \quad \lim_{t\uparrow\omega} \psi_s^2(t)\Big[r_s(t) + 2 + \frac{\pi_\omega(t) J_{\varphi_s}''(t)}{J_{\varphi_s}'(t)}\Big] = 0,$$
$$\lim_{t\uparrow\omega} \frac{\psi_s(t)}{E_s(t)} = 0 \ \text{ at } \ \gamma_s = 0, \quad \lim_{t\uparrow\omega} \psi_s^2(t) \sum_{\substack{i=1\\ i\neq s}}^{m} \frac{p_i(t)\varphi_i(H_s^{-1}(-\alpha_s J_{3s}(t)))}{p_s(t)\varphi_s(H_s^{-1}(-\alpha_s J_{3s}(t)))} = 0,$$

*the differential equation* (1) *has a $P_\omega(Y_0,0)$ – solution with asymptotic representations*

$$y(t) = H_s^{-1}(-\alpha_s J_{3s}(t))\Big[1 + \frac{o(1)}{G_s(t)\psi_s(t)}\Big] \ \text{ at } \ t\uparrow\omega,$$
$$y'(t) = -\alpha_s\pi_\omega(t)p_{0s}(t)\varphi_s\big(H_s^{-1}(-\alpha_s J_{3s}(t))\big)\big[1 + |E_s(t)|^{-\frac12}\psi_s^{-1}(t) o(1)\big] \ \text{ at } \ t\uparrow\omega.$$

*Moreover, there exists a two-parameter family of such solutions in case when $\gamma_s \in (0,1/3)$ or $\gamma_s = 0$ and $\alpha_s\nu_1 = 1$.*

# References


[1] V. M. Evtukhov and N. P. Kolun, Asymptotic representations of solutions of differential equations with regularly and rapidly varying nonlinearities. (Russian) *Mat. met. i fiz.-mat. polya* **60** (2017), no. 1, 32–43.

[2] V. M. Evtukhov and N. P. Kolun, Rapidly varying solutions of a second-order differential equation with regularly and rapidly varying nonlinearities. J. Math. Sci. (N.Y.) **235** (2018), no. 1, 15–34.

[3] V. M. Evtukhov and N. P. Kolun, Asymptotic of solutions of differential equations with regularly and rapidly varying nonlinearities. (Russian) *Neliniĭni Kolyvannya* **21** (2018), no. 3, 323–346.

[4] V. Marić, *Regular Variation and Differential Equations*. Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.