On One System of Nonlinear Partial Integro-Differential Equations with Source Terms

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Let us consider the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial U}{\partial x}\Big] + f(U) = 0, \quad \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big[a(S)\,\frac{\partial V}{\partial x}\Big] + f(V) = 0, \tag{1}$$

where

$$S(x,t) = 1 + \int\limits_0^t \Big[\Big(\frac{\partial U}{\partial x}\Big)^2 + \Big(\frac{\partial V}{\partial x}\Big)^2\Big]\,d\tau$$

and $a = a(S)$, $f = f(U)$ and $f = f(V)$ are given functions, constraints on which will be specified later.

The above-mentioned system with source terms is based on the well-known system of Maxwell's equations [12] by reducing it to the following integro-differential model [4]

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\bigg[a\bigg(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\bigg)\operatorname{rot} H\bigg], \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field.

In the rectangle $[0,1]\times[0,\infty]$ let us consider the following initial-boundary value problem with mixed boundary conditions:

$$U(0,t) = \frac{\partial U(x,t)}{\partial x}\bigg|_{x=1} = V(0,t) = \frac{\partial V(x,t)}{\partial x}\bigg|_{x=1} = 0, \;\; t \ge 0, \tag{3}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \;\; x \in [0,1], \tag{4}$$

where U_0 and V_0 are given functions.

Study of the models of type (2) have begun in [4]. In that work, in particular, based on Galerkin's modified method and compactness arguments as in [14, 18] for nonlinear parabolic equations the theorems of existence of a solution of the initial-boundary value problem with first kind boundary conditions for scalar and one-dimensional space case when $a(S) = 1 + S$ and uniqueness for more general cases are proven. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \le 1$ is studied in [2]. Asymptotic behavior as $t \to \infty$ of solutions of initial-boundary value problems for (2) type models are studied in [3, 6, 7, 9, 13, 16] and in a number of other works as well. In those works main attention is paid to one-dimensional cases. Finite element analogues and Galerkin's method algorithm as well as construction and investigation of semi-discrete and finite difference schemes for (2) type one-dimensional integro-differential models are studied in [1, 5, 7–11, 13, 15–17] and in other works as well for the linear case of diffusion coefficient.

The following statement is true [5, 8].

Theorem 1. *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, f is positively defined and monotonically increased function, $U_0, V_0 \in H^1(0,1)$, $U_0(0) = \frac{dU_0(x)}{dx}\big|_{x=1} = V_0(0) = \frac{dV_0(x)}{dx}\big|_{x=1} = 0$, and problem (1), (3), (4) has a solution, then it is unique and exponential stabilization of solution as $t \to \infty$ takes place.*

On $[0,1] \times [0,T]$, where T is a positive constant, let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \dots, M$; $j = 0, 1, \dots, N$ with $h = 1/M$, $\tau = T/N$ and let us consider the finite discrete scheme for problem (1), (3), (4):

$$\begin{gathered}
\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{a\Big(\tau \sum_{k=1}^{j+1} \big[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\big]\Big) u_{\bar{x},i}^{j+1}\bigg\}_x + f(u_i^{j+1}) = 0, \\
\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{a\Big(\tau \sum_{k=1}^{j+1} \big[(u_{\bar{x},i}^k)^2 + (v_{\bar{x},i}^k)^2\big]\Big) v_{\bar{x},i}^{j+1}\bigg\}_x + f(v_i^{j+1}) = 0, \\
i = 1, 2, \dots, M-1; \;\; j = 0, 1, \dots, N-1, \\
u_0^j = u_{\bar{x},M}^j = v_0^j = v_{\bar{x},M}^j = 0, \;\; j = 0, 1, \dots, N, \\
u_i(0) = U_{0,i}, \quad v_i(0) = V_{0,i}, \;\; i = 0, 1, \dots, M,
\end{gathered} \tag{5}$$

where the well-known notations of forward and backward derivatives are used.

Applying the u_i^{j+1} and v_i^{j+1} multiplicators for the first and second equations of system (5) respectively, it is not difficult to get the inequalities:

$$\|u^n\|^2 + \tau h \sum_{j=1}^{n} \sum_{i=1}^{M} (u_{i,\bar{x}}^j)^2 < C, \quad \|v^n\|^2 + \tau h \sum_{j=1}^{n} \sum_{i=1}^{M} (v_{i,\bar{x}}^j)^2 < C, \;\; n = 1, 2, \dots, N. \tag{6}$$

Here and in what follows C is a positive constant independent of τ and h.

The a priori estimates (6) guarantee the global solvability of problem (5).

The following statement is true.

Theorem 2. *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, f is positively defined and monotonically increased function and problem (1), (3), (4) has a sufficiently smooth solution, then the solution of problem (5) tends to the solution of the continuous problem (1), (3), (4) as $h \to 0$, $\tau \to 0$ and the following estimates are true:*

$$\|u^j - U^j\| \leq C(\tau + h), \quad \|v^j - V^j\| \leq C(\tau + h).$$

We have carried out numerous numerical experiments for problem (1), (3), (4) with different kinds of right hand sides and initial-boundary conditions. The obtained numerical results are in accordance to the theoretical findings.

References

[1] F. Chen, Crank–Nicolson fully discrete H^1-Galerkin mixed finite element approximation of one nonlinear integrodifferential model. *Abstr. Appl. Anal.* **2014**, Art. ID 534902, 8 pp.

[2] T. A. Dzhangveladze, The first boundary value problem for a nonlinear equation of parabolic type. (Russian) *Dokl. Akad. Nauk SSSR* **269** (1983), no. 4, 839–842; translation in *Soviet Phys. Dokl.* **28** (1983), 323–324.

[3] T. A. Dzhangveladze and Z. V. Kiguradze, Asymptotic behavior of the solution of a nonlinear integrodifferential diffusion equation. (Russian) *Differ. Uravn.* **44** (2008), no. 4, 517–529; translation in *Differ. Equ.* **44** (2008), no. 4, 538–550.

[4] D. G. Gordeziani, T. A. Dzhangveladze, and T. K. Korshia, Existence and uniqueness of the solution of a class of nonlinear parabolic problems. (Russian) *Differentsial'nye Uravneniya* **19** (1983), no. 7, 1197–1207; translation in *Differ. Equations* **19** (1984), 887–895.

[5] F. Hecht, T. Jangveladze, Z. Kiguradze and O. Pironneau, Finite difference scheme for one system of nonlinear partial integro-differential equations. *Appl. Math. Comput.* **328** (2018), 287–300.

[6] T. Jangveladze, On one class of nonlinear integro-differential parabolic equations. *Semin. I. Vekua Inst. Appl. Math. Rep.* **23** (1997), 51–87.

[7] T. Jangveladze, Convergence of a difference scheme for a nonlinear integro-differential equation. *Proc. I. Vekua Inst. Appl. Math.* **48** (1998), 38–43.

[8] T. Jangveladze, Z. Kiguradze and M. Kratsashvili, Uniqueness of solution ad fully discrete scheme to nonlinear integro-differential averaged model with source terms. *Miskolc Math. Notes* (accepted).

[9] T. Jangveladze, Z. Kiguradze and B. Neta, *Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations.* Elsevier/Academic Press, Amsterdam, 2016.

[10] T. Jangveladze, Z. Kiguradze, B. Neta and S. Reich, Finite element approximations of a nonlinear diffusion model with memory. *Numer. Algorithms* **64** (2013), no. 1, 127–155.

[11] Z. Kiguradze, Convergence of finite difference scheme and uniqueness of a solution for one system of nonlinear integro-differential equations with source terms. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2017*, Tbilisi, Georgia, December 24–26, pp. 102–105; `http://www.rmi.ge/eng/QUALITDE-2017/Kiguradze_Z_workshop_2017.pdf`.

[12] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media.* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957.

[13] H.-l. Liao and Y. Zhao, Linearly localized difference schemes for the nonlinear Maxwell model of a magnetic field into a substance. *Appl. Math. Comput.* **233** (2014), 608–622.

[14] J.-L. Lions, *Quelques Méthodes de Résolution Des Problèmes Aux Limites Non Linéaires.* (French) Dunod, Gauthier-Villars, Paris, 1969.

[15] N. Sharma, M. Khebchareon, K. Sharma, and A. K. Pani, Finite element Galerkin approximations to a class of nonlinear and nonlocal parabolic problems. *Numer. Methods Partial Differential Equations* **32** (2016), no. 4, 1232–1264.

[16] N. Sharma and K. K. Sharma, Unconditionally stable numerical method for a nonlinear partial integro-differential equation. *Comput. Math. Appl.* **67** (2014), no. 1, 62–76.

[17] N. Sharma and K. K. Sharma, Finite element method for a nonlinear parabolic integro-differential equation in higher spatial dimensions. *Appl. Math. Model.* **39** (2015), no. 23-24, 7338–7350.

[18] M. I. Vishik, Solubility of boundary-value problems for quasi-linear parabolic equations of higher orders. (Russian) *Mat. Sb. (N.S.)* **59 (101)** (1962), 289–325.