Periodic Solutions of Higher Order Nonlinear Hyperbolic Equations

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Let $m_1,\dots,m_n$ be positive integers. Consider the periodic problem

$$u^{(\mathbf{m})}=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big), \tag{1}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_i)=u(\mathbf{x})\ \ (i=1,\dots,n). \tag{2}$$

Here $\mathbf{x}=(x_1,\dots,x_n)$, $\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$, $\boldsymbol{\omega}_\mathbf{i}=(0,\dots,\omega_i,\dots,0)$, $\mathbf{m}=(m_1,\dots,m_n)$ is a multi-index,

$$u^{(\mathbf{m})}(\mathbf{x})=\frac{\partial^{m_1+\cdots+m_n}u(\mathbf{x})}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\,,$$

$\mathcal{D}^{\mathbf{m}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}\le\mathbf{m}}$, $\widetilde{\mathcal{D}}^{\mathbf{m}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}<\mathbf{m}}$, $f\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m+1}})$ and $C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m+1}})$ is the space of continuous functions $v(\mathbf{x},\mathbf{Z})$ that are $\boldsymbol{\omega}$-periodic with respect to the variable $\mathbf{x}$, i.e.

$$v(\mathbf{x}+\boldsymbol{\omega}_i,\mathbf{Z})=v(\mathbf{x},\mathbf{Z})\ \ (i=1,\dots,n).$$

By a solution of problem (1),(2) we understand a classical solution, i.e., a function $u\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ satisfying equation (1) everywhere in $\mathbb{R}^n$.

Problems on doubly periodic solutions for hyperbolic equations of the second and fourth orders were studied in [1–3]. Problem (1), (2) for the case $n>2$ remained virtually unstudied until recently. The linear case of problem (1), (2) was investigated in [4].

Throughout the paper the following notations will be used:

$\mathbf{m}=(m_1,\dots,m_n)$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$.

$\mathbb{R}^{\boldsymbol{\alpha}}=\mathbb{R}^{\alpha_1\times\cdots\times\alpha_n}$.

$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)<\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\iff\alpha_i\le\beta_i\ (i=1,\dots,n)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\le\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\mathbf{0}=(0,\dots,0)$, $\mathbf{1}=(1,\dots,1)$, $\mathbf{1}_i=(0,\dots,0,1,0,\dots,0)$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i\ \ \alpha_i>0\}$, $\|\boldsymbol{\alpha}\|=|\alpha_1|+\cdots+|\alpha_n|$.

$\boldsymbol{\Upsilon}_\mathbf{m}=\{\boldsymbol{\alpha}<\mathbf{m}:\ \alpha_i=m_i$ for some $i\in\{1,\dots,n\}\}$.

$\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$, $\boldsymbol{\omega}_\mathbf{i}=(0,\dots,\omega_i,\dots,0)$.

$\Omega=[0,\omega_1]\times\cdots\times[0,\omega_n]$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)x_1,\dots,\chi(\alpha_n)x_n)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$. $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\alpha}$.

$\mathbf{Z}=(z_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}<\mathbf{m}}$; $f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})=\frac{\partial f(\mathbf{x},\mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}$.

The variables $z_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_\mathbf{m}$) are called ***principal phase variables*** of the function $f(\mathbf{x},\mathbf{Z})$.

$C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u:\Omega\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ ($\boldsymbol{\alpha}\le\mathbf{m}$), endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. functions that are ω_i-periodic with respect to the variable x_i $(i=1,\dots,n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. functions that are ω_i-periodic with respect to the variable x_i $(i=1,\dots,n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

If $z_0\in\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $r>0$, then

$$\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(z_0;r)=\Big\{z\in\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n):\ \|z-z_0\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}}\le r\Big\}.$$

$C^{\mathbf{m},k}_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\beta}})$ the space of continuous functions $v(\boldsymbol{x},\mathbf{Z})$ such that $v(\,\cdot\,,\mathbf{Z})\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ for every $\mathbf{Z}\in\mathbb{R}^{\boldsymbol{\beta}}$ and $v(\mathbf{x},\,\cdot\,)\in C^k(\mathbb{R}^{\boldsymbol{\beta}})$ for every $\mathbf{x}\in\mathbb{R}^n$.

Let $p_{0\,\boldsymbol{\alpha}}\in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}<\boldsymbol{m})$ and let $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ be an arbitrary function. Along with the equation (1) consider the following equations

$$u^{(\mathbf{m})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}),\tag{3}$$

$$u^{(\mathbf{m})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x})u^{(\boldsymbol{\alpha})},\tag{4}$$

and

$$u^{(\mathbf{m})}=(1-\lambda)\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{0\,\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+\lambda f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big),\tag{5}$$

where $\lambda\in[0,1]$, $p_{\boldsymbol{\alpha}}[z](\mathbf{x})=f_{\boldsymbol{\alpha}}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[z](\mathbf{x}))$, and

$$p_{\lambda\,\boldsymbol{\alpha}}[z](\mathbf{x})=(1-\lambda)p_{0\,\boldsymbol{\alpha}}(\mathbf{x})+\lambda p_{\boldsymbol{\alpha}}[z](\mathbf{x}).$$

Definition 1. Let the function $f(\mathbf{x},\mathbf{Z})$ be continuously differentiable with respect to the phase variables $\mathbf{v}$. We say that problem (1), (2) to is *strongly* (u_0,r)-*well-posed*, if:

(I) it has a solution $u_0(\mathbf{x})$;

(II) in the neighborhood $\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0;r)$ u_0 is the unique solution;

(III) there exists $\varepsilon_0>0$, $\delta_0>0$ and $M_0>0$ such that for any $\delta\in(0,\delta_0)$, and $\widetilde{f}(\mathbf{x},\mathbf{Z})$ satisfying the inequalities

$$\sum_{\boldsymbol{\alpha}<\mathbf{m}}\big|f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})-\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})\big|<\varepsilon_0,\tag{6}$$

$$\big|f(\mathbf{x},\mathbf{Z})-\widetilde{f}(\mathbf{x},\mathbf{Z})\big|<\delta\tag{7}$$

in the neighborhood $\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0;r)$ the problem

$$u^{(\mathbf{m})}=\widetilde{f}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]),$$
$$u(\mathbf{x}+\boldsymbol{\omega}_i)=u(\mathbf{x})\ \ (i=1,\dots,n)$$

has a unique solution $\widetilde{u}$ and

$$\|u-\widetilde{u}\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}<M_0\delta.$$

Definition 2. Problem (1), (2) is called *strongly well-posed* if it is strongly (u_0,r)-well-posed for every $r>0$.

Theorem 1. *Let the function $f(\mathbf{x},Z)$ be continuously differentiable with respect to the phase variables, and let there exist a positive number M_0 such that*

$$|f_{\boldsymbol{\alpha}}(\mathbf{x},Z)|\le M_0\ \text{ for }\ (\mathbf{x},Z)\in\mathbb{R}^n\times\mathbb{R}^{\mathbf{m+1}}.$$

Furthermore, let for arbitrary $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $\lambda\in[0,1)$ problem (3), (2) be well–posed and its solution u_λ admit the estimate

$$\|u_\lambda\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M\|q\|_{C_{\boldsymbol{\omega}}},$$

where M is a positive number independent of λ, z and q. Then problem (1), (2) is strongly well–posed.

Consider the "perturbed" equation

$$u^{(\mathbf{m})}=f(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])+q(\mathbf{x},\mathcal{D}^{\mathbf{m\text{-}1}}[u]). \tag{8}$$

Theorem 2. *Let the function f satisfy all of the conditions of Theorem 1, and let $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ be such that*

$$\lim_{\|\mathbf{Z}\|\to+\infty}\frac{|q(\mathbf{x},\mathbf{Z})|}{\|\mathbf{Z}\|}=0 \tag{9}$$

uniformly on $\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}}$. Then problem (8), (2) has at least one solution

Theorem 3. *Let the function $f(\mathbf{x},Z)$ be continuously differentiable with respect to the phase variables, and let there exist a positive number M and a nondecreasing continuous function $\eta:[0,+\infty)\to[0,+\infty)$, $\eta(0)=0$ such that:*

(i) *for every $\lambda\in[0,1)$ an arbitrary solution u_λ of problem (5), (2) admits the estimates*

$$u_\lambda\in\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(0;M),\ \ \|w_{\lambda\,\delta}\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le\eta(|\delta|),$$

where $w_{\lambda\,\delta}(\mathbf{x})=u_\lambda(\mathbf{x}+\delta)-u_\lambda(\mathbf{x})$;

(ii) *problem (4), (2) is well–posed for every $\lambda\in[0,1)$ and $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, $\|z\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M$;*

(iii) *problem (4), (2) has only the trivial solution for $\lambda=1$ and arbitrary $z\in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, $\|z\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}\le M$.*

Then problem (1), (2) has a solution $u_0\in\widetilde{\mathcal{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(0;M)$, and it is strongly strongly *(u_0,r) well-posed for some $r>0$.*

Consider the equations of even and odd orders:

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}}[u]\big)u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q(\mathbf{x},\mathcal{D}^{\mathbf{m-1}}[u]), \tag{10}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \Big(p_{\boldsymbol{\alpha}}\big(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}}[u]\big)\ u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\alpha})} + q(\mathbf{x},\mathcal{D}^{\mathbf{m-1}}[u]) \tag{11}$$

and

$$u^{(2\mathbf{m}+\mathbf{1}_n)} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} (p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}+\mathbf{1}_n}[u]))^{(\boldsymbol{\beta})} + \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(2\boldsymbol{\alpha})} + q(\mathbf{x},\mathcal{D}^{\mathbf{m-1}}[u]). \tag{12}$$

Theorem 4. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha},\boldsymbol{\beta}\le\boldsymbol{m})$, *and let* $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ *satisfy equality* (9) *uniformly on* $\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}}$. *Furthermore, let there exist* $\delta>0$ *such that*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},\mathbf{Z})v_{\boldsymbol{\alpha}}v_{\boldsymbol{\beta}} \ge \delta \sum_{\boldsymbol{\alpha}\le\mathbf{m}} v^2_{\boldsymbol{\alpha}} \ \ \textit{for}\ \ (\mathbf{x},\mathbf{Z})\in\mathbb{R}^n\times\mathbb{R}^{2\boldsymbol{m}+\mathbf{1}}.$$

Then problem (10), (2) *has at least one solution.*

Corollary 1. *Let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha}\le\boldsymbol{m})$, *and let* $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ *satisfy equality* (9) *uniformly on* $\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}}$. *Furthermore, let there exist* $\delta>0$ *such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z}) \ge \delta \ \ \textit{for}\ \ (\mathbf{x},\mathbf{Z})\in\mathbb{R}^n\times\mathbb{R}^{2\boldsymbol{m}+\mathbf{1}} \ \ (\boldsymbol{\alpha}\le\boldsymbol{m}).$$

Then problem (11), (2) *has at least one solution.*

Theorem 5. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\alpha}+\mathbf{1}})$ $(\boldsymbol{\alpha},\boldsymbol{\beta}\le\boldsymbol{m})$, *and let* $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ *satisfy equality* (14) *uniformly on* $\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}}$. *Furthermore, let there exist* $\delta>0$ *such that*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x},\mathbf{Z})z_{\boldsymbol{\alpha}}\,z_{\boldsymbol{\beta}} \ge \delta \sum_{\boldsymbol{\alpha}\le\mathbf{m}} z^2_{\boldsymbol{\alpha}} \ \ \textit{for}\ \ (\mathbf{x},\mathbf{Z})\in\mathbb{R}^n\times\mathbb{R}^{2\boldsymbol{m}+\mathbf{1}}$$

and

$$(-1)^{\|\boldsymbol{\alpha}\|}\sigma p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \ge \delta \ \ \textit{for}\ \ \mathbf{x}\in\mathbb{R}^n \ \ (\boldsymbol{\alpha}\le\boldsymbol{m}).$$

Then problem (12), (2) *has at least one solution.*

Remark 1. In Theorems 1–3 continuous differentiability of the function $f(\mathbf{x},\mathbf{Z})$ with respect to the phase variables $\mathbf{Z}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorems more cumbersome. However, Lipschitz continuity of the function $f(\mathbf{x},\mathbf{Z})$ with respect to the *principal* phase variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{m}})$ is essential and cannot be replaced by Hölder continuity with the exponent $\gamma\in(0,1)$.

As an example consider the two–dimensional problem

$$u^{(2,2)} = u^{(2,0)} + u^{(0,2)} - \delta^{1-\gamma}|u^{(0,2)}-u|^{\gamma}\operatorname{sgn}(u^{(0,2)}-u) - u - \sin x_2, \tag{13}$$

$$u(x_1+2\pi,x_2) = u(x_1+2\pi,x_2), \quad u(x_1,x_2+2\pi) = u(x_1,x_2) \tag{14}$$

where $\delta\ge0$ and $\gamma\in(0,1)$.

Let u be a solution of problem (10), (11). Set:

$$v(x_1,x_2) = u^{(0,2)}(x_1,x_2) - u(x_1,x_2). \tag{15}$$

Then v is a solution of the problem

$$v^{(2,0)} = v - \delta^{1-\gamma}|v|^{\gamma}\operatorname{sgn}(v) - \sin x_2, \tag{16}$$

$$v(x_1 + 2\pi, x_2) = v(x_1, x_2). \tag{17}$$

If $\delta = 0$, then it is clear that problem (16), (17) is a uniquely solvable linear periodic problem with the solution

$$v(x_1, x_2) \equiv \sin x_2,$$

and problem (10), (11) is a well–posed linear problem with the solution

$$u(x_1, x_2) \equiv u(x_2) = \int\limits_{x_2-2\pi}^{x_2} \frac{\cosh(x_2 - t - \pi)}{2\sinh(\pi)}\sin t\,dt.$$

Let us show that problem (10), (11) has no classical solutions for sufficiently small $\delta > 0$. For that it is sufficient to show that for sufficiently small $\delta > 0$ problem (16), (17) has no solution that is continuous with respect to x_2.

Problem (16), (17) is a periodic problem for an ordinary differential equation depending on the parameter x_2. It has a solution $v(x_1, x_2) \equiv v^*(x_2)$, where, for every x_2, $v^*(x_2)$ is the root of the equation

$$v - \delta^{1-\gamma}|v|^{\gamma}\operatorname{sgn}(v) - \sin x_2 = 0. \tag{18}$$

One can easily show that problem (16), (17) is solvable for every x_2 if $\delta \in (0, 1)$. Moreover, if $\delta \in (0, 2^{\frac{1}{\gamma-1}})$, then problem (16), (17) is uniquely solvable for $x_2 = \frac{\pi}{2}$, and its solution is positive. The latter fact implies that $v^*(\frac{\pi}{2}) > \delta$.

Let $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$, and let $v(x_1, x_2)$ be a solution of problem (16), (17) that is a continuous function of x_2. Then $v(x_1, \frac{\pi}{2}) = v^*(\frac{\pi}{2}) > \delta$. Due to continuity there exists $\varepsilon > 0$ such that

$$v(x_1, x_2) \geq \delta \ \text{ for } \ x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{19}$$

But then problem (16), (17) is uniquely solvable for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Indeed, let $v_1(x_1) \geq \delta$ and $v_2(x_1) \geq \delta$ be arbitrary solutions of problem (16),(17) for some $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Then $v(x_1) = v_2(x_1) - v_1(x_1)$ is a solution of the problem

$$v'' = (1 - \theta(x_1, x_2))v, \quad v(x_1 + 2\pi) = v(x_1), \tag{20}$$

where

$$\theta(x_1, x_2) = \gamma \int\limits_0^1 \frac{\delta^{1-\gamma}}{(v_1(x_1, x_2) + (1-t)(v_2(x_1, x_2) - v_1(x_2, x_1)))^{1-\gamma}}\,dt \leq \gamma < 1. \tag{21}$$

The latter inequality implies that problem (20) has only the trivial solution, i.e. problem (16), (17) is uniquely solvable. Consequently, $v(x_1, x_2) = v^*(x_2)$ for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. However, it is easy to see that a positive root of equation (18) is strictly bigger than δ for $x_2 \in (0, \pi)$. Hence

$$v(x_1, x_2) = v^*(x_2) > \delta \ \text{ for } \ x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{22}$$

From (19)–(22) one can easily deduce that

$$v(x_1, x_2) = v^*(x_2) > \delta \ \text{ for } \ x_2 \in (0, \pi). \tag{23}$$

Similarly one can show that

$$v(x_1,x_2)=v^*(x_2)<-\delta \ \text{ for } \ x_2\in(-\pi,0). \tag{24}$$

(23) and (24) imply that $v^*(0+)=\delta$ and $v^*(0-)=-\delta$. Thus $v(x_1,x_2)\equiv v^*(x_2)$ is discontinuous at 0. Consequently, problem $(13),(14)$ has no classical solutions for sufficiently small $\delta\in(0,2^{\frac{1}{\gamma-1}})$.

This is the result of the fact that the righthand side of equation (13) is not Lipschitz continuous with respect to the principal phase variables, but instead is a Hölder continuous function with the exponent $\gamma\in(0,1)$.

Remark 2. The aforementioned example also demonstrates that:

(A) Condition (6) in Definition 1 is optimal and cannot be relaxed;

(B) Only inequality (7), without inequality (6) does not guarantee even solvability of a perturbed problem.

References

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