The Dirichlet Problem for Second Order Essentially Singular Ordinary Differential Equations

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On a finite open interval $]a,b[$, we consider the differential equation

$$u'' = f(t,u) \tag{1}$$

with the Dirichlet boundary conditions

$$u(a+) = 0, \quad u(b-) = 0, \tag{2}$$

where $f :]a,b[\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $u(a+)$ and $u(b-)$ are, respectively, the right and the left limits of the function u at the points a and b.

We are interested in the case where the function f has a nonintegrable singularity in the time variable at the points a and b.

In the earlier known theorems of the existence and uniqueness of a solution of the singular boundary value problem $(1),(2)$ it was assumed that

$$\int\limits_a^b (t-a)(b-t)|f(t,0)|\,dt < +\infty$$

(see, e.g., [1–9] and the references therein). Unlike them, the results below cover the case when for arbitrary $x \in \mathbb{R}$ and $\ell > 0$ the condition

$$\int\limits_a^b (t-a)^\ell (b-t)^\ell |f(t,x)|\,dt = +\infty \tag{3}$$

is fulfilled. The results are new also for the linear differential equation

$$u'' = p(t)u + q(t), \tag{4}$$

where p and $q :]a,b[\to\mathbb{R}$ are continuous functions with singularities at the points a and b.

We use the following notation.

$\mathbb{R}$ is the set of real numbers, $[x]_- = \frac{|x|-x}{2}$.

Definition 1. The linear homogeneous differential equation

$$u'' = p(t)u \tag{4_0}$$

with continuous coefficients $p :]a,b[\to\mathbb{R}$ is said to be ***nonoscillatory in the interval*** $[a,b]$ if every its nontrivial solution, satisfying the initial condition

$$u(a+) = 0,$$

satisfies also the inequalities

$$u(t) \neq 0 \ \text{ for } \ a < t < b, \quad \liminf_{t\to b} |u(t)| > 0.$$

Definition 2. The function $G:]a,b[\times]a,b[\to\mathbb{R}$ is said to be ***Green's function of problem*** (4_0)**,** (2) if for every $s\in]a,b[$ the function $u(t)=G(t,s)$ is continuous in the interval $]a,b[$ and satisfies the boundary conditions (2), while the restrictions of u to $]a,s[$ and $]s,b[$ are the solutions of equation (4_0) and

$$u'(s+)-u'(s-)=1.$$

If G is Green's function of problem $(4_0),(2)$, we put

$$H(p)(s)=\sup\big\{|G(t,s)|:\ a<t<b\big\}\ \text{ for }\ a<s<b.$$

Proposition 1. *If*

$$\int\limits_a^b (t-a)(b-t)[p(t)]_-\,dt<+\infty \tag{5}$$

and the homogeneous problem $(4_0),(2)$ *has only the trivial solution, then there exists a unique Green's function of that problem, and*

$$\sup\Big\{\frac{H(p)(s)}{(s-a)(b-s)}\ :\ a<s<b\Big\}<+\infty.$$

Theorem 1. *If the homogeneous problem* $(4_0),(2)$ *has only the trivial solution and along with* (5) *the condition*

$$\int\limits_a^b H(p)(t)|q(t)|\,dt<+\infty \tag{6}$$

is fulfilled, then problem $(4),(2)$ *is uniquely solvable and its solution admits the representation*

$$u(t)=\int\limits_a^b G(t,s)q(s)\,ds\ \ \textit{for}\ \ a<t<b, \tag{7}$$

where G *is Green's function of problem* $(4_0),(2)$.

Corollary 1. *Let there exist a nondecreasing in some right neighbourhood of the point* a *and a nonincreasing in some left neighbourhood of the point* b *continuously differentiable function* $\delta:]a,b[\to]0,+\infty[$ *such that*

$$\delta(a+)=\delta'(a+)=0,\quad \delta(b-)=\delta'(b-)=0,$$
$$\liminf_{t\to a}(\delta^2(t)p(t))>0,\quad \liminf_{t\to b}(\delta^2(t)p(t))>0.$$

If, moreover,

$$\int\limits_a^b (t-a)(b-t)[p(t)]_-\,dt\le b-a,\quad \int\limits_a^b \delta(t)|q(t)|\,dt<+\infty,$$

then problem $(4),(2)$ *is uniquely solvable and its solution admits representation* (7).

Remark 1. Green's formula (7) has been derived earlier only in the case, where

$$\int\limits_a^b (t-a)(b-t)|p(t)|\,dt<+\infty,\quad \int\limits_a^b (t-a)(b-t)|q(t)|\,dt<+\infty$$

(see [6, Theorem 1.1]), but Theorem 1 covers the case in which these functions have at the points a and b singularities of infinite order. Indeed, if

$$\delta(t) \equiv \exp\Big(-\frac{1}{t-a}-\frac{1}{b-t}\Big),$$
$$p(t) \equiv p_0(t)\delta^{-2}(t), \quad q(t) \equiv q_0(t)\delta^{-1}(t),$$

where $p_0 :]a,b[\to]1,+\infty[$, $q_0 : [a,b] \to [1,+\infty[$ are arbitrary continuous functions, then for any $\ell > 0$, the equalities

$$\int_a^b (t-a)^\ell (b-t)^\ell p(t)\,dt = +\infty, \quad \int_a^b (t-a)^\ell (b-t)^\ell q(t)\,dt = +\infty$$

are fulfilled. Nevertheless, according to Corollary 1, problem $(4),(2)$ is uniquely solvable and its solution admits representation (7).

Theorem 2. *Let on the set* $]a,b[\,\times\mathbb{R}$ *the inequality*

$$f(t,x)\operatorname{sgn}(x) \geq p(t)|x| + q(t) \tag{8}$$

be fulfilled, where $p :]a,b[\to \mathbb{R}$ *and* $q :]a,b[\to]-\infty,0]$ *are continuous functions. If, moreover, the homogeneous equation* (4_0) *is nonoscillatory and conditions* (5) *and* (6) *hold, then problem* $(1),(2)$ *has at least one solution.*

Corollary 2. *Let on the set* $]a,b[\,\times\mathbb{R}$ *inequality* (8) *be fulfilled, where* $p :]a,b[\to \mathbb{R}$ *and* $q :]a,b[\to [0,+\infty[$ *are continuous functions and, in addition,* p *is continuously differentiable and nonincreasing (nondecreasing) in some right neighbourhood of the point* a *(in some left neighbourhood of the point* b*). If, moreover,*

$$p(a+) = +\infty, \ \lim_{t\to a}\big(p^{-3/2}(t)p'(t)\big) = 0, \quad p(b-) = +\infty, \ \lim_{t\to b}\big(p^{-3/2}(t)p'(t)\big) = 0,$$
$$\int_a^b (t-a)(b-t)[p(t)]_-\,dt \leq b-a, \quad \int_a^b \frac{|q(t)|}{\sqrt{1+|p(t)|}}\,dt < +\infty,$$

then problem $(1),(2)$ *has at least one solution.*

Theorem 3. *Let on the set* $]a,b[\,\times\mathbb{R}$ *the condition*

$$\big(f(t,x)-f(t,y)\big)\operatorname{sgn}(x-y) \geq p(t)|x-y| \tag{9}$$

be fulfilled, where $p :]a,b[\to \mathbb{R}$ *is a continuous function satisfying condition* (5)*. If, moreover, the homogeneous equation* (4_0) *is nonoscillatory and*

$$\int_a^b H(p)(t)|f(t,0)|\,dt < +\infty,$$

then problem $(1),(2)$ *has one and only one solution.*

Corollary 3. *Let on the set* $]a,b[\,\times\mathbb{R}$ *condition* (9) *be fulfilled, where* $p :]a,b[\to \mathbb{R}$ *is a function satisfying the conditions of Corollary* 2*. If, moreover,*

$$\int_a^b \frac{|f(t,0)|}{\sqrt{1+|p(t)|}}\,dt < +\infty,$$

then problem $(1),(2)$ *has one and only one solution.*

Example 1. Let

$$f(t,x)=\sum_{k=1}^{n}p_k(t)|x|^{\lambda_k}\operatorname{sgn}x+p_0(t)\exp\Big(\frac{2}{t-a}+\frac{2}{b-t}\Big)u+q_0(t)\exp\Big(\frac{1}{t-a}+\frac{1}{b-t}\Big),$$

where $p_k:\,]a,b[\,\to[0,+\infty[$ $(k=1,\dots,n)$, $p_0:\,]a,b[\,\to[1,+\infty[$, $q_0:[a,b]\to[1,+\infty[$ are continuous function, $\lambda_k=const>0$ $(k=1,\dots,n)$. Then for arbitrary $x\in\mathbb{R}$ and $\ell>0$ condition (3) is fulfilled. On the other hand, according to Corollary 3, problem (1), (2) has one and only one solution.

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